dynamic mass damper

Edwards, Penney, Calvis DiffEq and LinAlg Editions 1-3: 7.4.14,
Edwards, Penney and Calvis, Differential Equations and Linear Algebra Edition 4: 7.5.14.

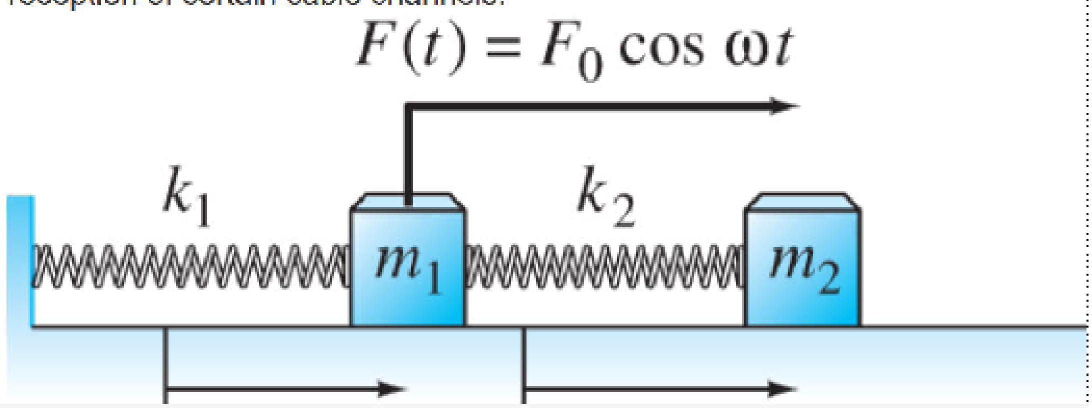
Do a web search or check Wikipedia for the description of a **dynamic oscillator damper**. [wiki, DMD]

This a toy model of coupling two oscillators in a driven system so that one will remain at rest! We adjust the second small mass until we find that the first large mass (which is being driven by an oscillating applied force) does not move in the response mode to the applied oscillation in the steady state situation (ignoring the undriven motion of the system). This idea underlies the dynamic damping of the swaying motion of tall buildings, where one engineers an internal damping oscillator in the top of the building to counteract the natural swaying motion. This idea of a tuned mass damper also underlies part of the solution to the swaying Millenium Bridge.

problem

In the diagram, we can arrange that the second mass is moving towards the first mass when the external force is pushing that first mass to the right so that the second spring pushes back and the first mass does not move and vice versa when moving away. The "force" ***F*** refers to the force per unit mass since it has already been divided by the mass when the second derivatives were solved for.

14. In the system of Fig. 7.5.12, assume that $m_1 = 1,\ k_1 = 50,\ k_2 = 10,$ and $F_0 = 5$ in mks units, and that $\omega = 10$. Then find m_2 so that in the resulting steady periodic oscillations, the mass m_1 will remain at rest(!). Thus the effect of the second mass-and-spring pair will be to neutralize the effect of the force on the first mass. This is an example of a *dynamic damper*. It has an electrical analogy that some cable companies use to prevent your reception of certain cable channels.



$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -\frac{k1+k2}{m1} & \frac{k2}{m1} \\ \frac{k2}{m2} & -\frac{k2}{m2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 5\cos(10\,t) \\ 0 \end{bmatrix}$$

solution

Note the driving frequency of 10 radians/sec is about 1.6 seconds.

> $evalf\left(\frac{10}{2\pi}\right)\cdot Hz$

$$1.591549430\ Hz \tag{2.1}$$

The first mass has a natural frequency of $\omega_1 = \sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{50}{1}} \approx 7.07$. The second spring is a factor of 5 weaker $\left(k_2 = \frac{1}{5}\,k_1\right)$ so requires a much smaller mass to get its natural frequency in the same ball park. Not sure if this is relevant or not, but it smells right.

> $\sqrt{50.}$

$$7.071067812 \tag{2.2}$$

> $with(LinearAlgebra) : with(plots) :$

The 2 mass 3 spring Hookes law coefficient matrix is $\begin{bmatrix} -\frac{(k1+k2)}{m1} & \frac{k2}{m1} \\ \frac{k2}{m2} & -\frac{k2}{m2} \end{bmatrix}$ so with these parameters we get the coefficient matrix:

> $A := \begin{bmatrix} -\frac{50+10}{1} & \frac{10}{1} \\ \frac{10}{m2} & -\frac{10}{m2} \end{bmatrix}$

$$A := \begin{bmatrix} -60 & 10 \\ \frac{10}{m2} & -\frac{10}{m2} \end{bmatrix} \tag{2.3}$$

Notice that for the following mass value $m2 = \frac{1}{8}$, the frequency $\omega = 10$ becomes an eigenfrequency (corresponding to eigenvalue $\lambda = -\omega^2 = -100$)leading to resonance and the trial vector solution in the method of undetermined coefficients must be multiplied by t.

> $Determinant\left(A - \left(-10^2\right) IdentityMatrix(2)\right) = 0;\ m2 = solve(\%)$

$$\frac{500\,(-1 + 8\,m2)}{m2} = 0$$

$$m2 = \frac{1}{8} \tag{2.4}$$

> $subs\left(m2 = \frac{1}{8}, A\right); Eigenvectors(\%); evalf\left(\sqrt{40.}\right)$

$$\begin{bmatrix} -60 & 10 \\ 80 & -80 \end{bmatrix}$$

$$\left[\begin{array}{c} -40 \\ -100 \end{array}\right], \left[\begin{array}{cc} \frac{1}{2} & -\frac{1}{4} \\ 1 & 1 \end{array}\right]$$

$$6.324555320 \quad \textbf{(2.5)}$$

The larger eigenvalue corresponds to the eigenfrequency 10 of the coupled system,

The differential equations:

> $deqs := x_1''(t) + 60\, x_1(t) - 10\, x_2(t) = 5\cos(10\, t), x_2''(t) - \frac{10}{m2}\, x_1(t) + \frac{10}{m2}\, x_2(t) = 0$

$$deqs := D^{(2)}(x_1)(t) + 60\, x_1(t) - 10\, x_2(t) = 5\cos(10\, t), D^{(2)}(x_2)(t) - \frac{10\, x_1(t)}{m2} + \frac{10\, x_2(t)}{m2} = 0 \quad \textbf{(2.6)}$$

The prime notation will not allow simplifying the derivative of these substituted trial functions from the method of undetermined coefficients! so we use the explicit derivative

> $deqs2 := \frac{d^2}{dt^2} x1(t) + 60\, x1(t) - 10\, x2(t) - 5\cos(10\, t) = 0, \frac{d^2}{dt^2}\, x2(t) - \frac{10}{m2}\, x1(t) + \frac{10}{m2}\, x2(t) = 0;$

$dsolve(\{deqs2\}, \{x1(t), x2(t)\}) :$

$$deqs2 := \frac{d^2}{dt^2}\, x1(t) + 60\, x1(t) - 10\, x2(t) - 5\cos(10\, t) = 0, \frac{d^2}{dt^2}\, x2(t) - \frac{10\, x1(t)}{m2} + \frac{10\, x2(t)}{m2} = 0 \quad \textbf{(2.7)}$$

Backsubtituting these trial solutions into the DEs and simplifying leads to these equations:

> $factor(eval((subs(x1(t) = a\cos(10\, t), x2(t) = b\cos(10\, t), [deqs2]))))$

$$\left[-5\cos(10\, t)\,(8\, a + 2\, b + 1) = 0, -\frac{10\cos(10\, t)\,(10\, b\, m2 + a - b)}{m2} = 0\right] \quad \textbf{(2.8)}$$

The equations for determining the coefficients are then these:

> $Eqs := 8\, a + 2\, b + 1 = 0, 10\, b\, m2 + a - b = 0$

$$Eqs := 8\, a + 2\, b + 1 = 0, 10\, b\, m2 + a - b = 0 \quad \textbf{(2.9)}$$

We want a solution for which $a = 0$ so that the first mass does not move:

> $solparams := solve(\{Eqs\}, \{a, b\}); solve(\{Eqs, a = 0\}, \{a, b, m2\})$

$$solparams := \left\{a = -\frac{10\, m2 - 1}{10\,(-1 + 8\, m2)}, b = \frac{1}{10\,(-1 + 8\, m2)}\right\}$$

$$\left\{a = 0, b = -\frac{1}{2}, m2 = \frac{1}{10}\right\} \quad \textbf{(2.10)}$$

Notice that this solution 1/10 for the second mass is close to the value 1/8 , the latter of which leads to a natural frequency 10 which means it will result in a big response through resonance so that the much smaller mass can compete with the larger mass.

> $subs\left(m2 = \frac{1}{10.}, [deqs2]\right)$

$$\left[\frac{d^2}{dt^2}\, x1(t) + 60\, x1(t) - 10\, x2(t) - 5\cos(10\, t) = 0, \frac{d^2}{dt^2}\, x2(t) - 100.0000000\, x1(t) + 100.0000000\, x2(t) = 0\right] \quad \textbf{(2.11)}$$

Numerically, the general solution and the particular response function solution for this particular mass choice:

> $map\left(fnormal, evalf\left(dsolve\left(subs\left(m2=\frac{1}{10.}, [deqs2]\right), \{x1(t), x2(t)\}\right)\right)\right);$
$subs(\_C1=0, \_C2=0, \_C3=0, \_C4=0, \%)$

$$\{x1(t) = 0.5741657390\, c_1 \cos(6.525597760\, t) + 0.5741657390\, c_3 \sin(6.525597760\, t) - 0.1741657388\, c_2 \cos(10.83589285\, t) - 0.1741657388\, c_4 \sin(10.83589285\, t), x2(t) = 1.000000000\, c_1 \cos(6.525597760\, t) + 1.000000000\, c_3 \sin(6.525597760\, t) + 1.000000000\, c_2 \cos(10.83589285\, t) + 1.000000000\, c_4 \sin(10.83589285\, t) - 0.5000000000 \cos(10.\, t)\}$$

$$\{x1(t) = -0., x2(t) = -0.5000000000 \cos(10.\, t)\} \quad (2.12)$$

The first mass is stationary, while the second mass is in the oppositive direction from the applied force in order to cancel it out. The new eigenvfrequencies here in the above solution are:

> $subs\left(m2=\frac{1}{10}, A\right); \Lambda, B := Eigenvectors(\%); evalf([\%]); \begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix} = evalf(map(\mathrm{sqrt}, -\Lambda))$

$$\begin{bmatrix} -60 & 10 \\ 100 & -100 \end{bmatrix}$$

$$\Lambda, B := \begin{bmatrix} -80 + 10\sqrt{14} \\ -80 - 10\sqrt{14} \end{bmatrix}, \begin{bmatrix} \frac{10}{-20+10\sqrt{14}} & \frac{10}{-20-10\sqrt{14}} \\ 1 & 1 \end{bmatrix}$$

$$\left[\begin{bmatrix} -42.58342613 \\ -117.4165739 \end{bmatrix}, \begin{bmatrix} 0.5741657386 & -0.1741657387 \\ 1. & 1. \end{bmatrix}\right]$$

$$\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix} = \begin{bmatrix} 6.525597760 \\ 10.83589285 \end{bmatrix} \quad (2.13)$$

This method of undetermined coefficients will not work for the mass choice $m2 = \frac{1}{8}$ where the frequency $\omega = 10$ is a natural frequency of the system, which leads to resonance and a trial solution multiplied by t.

Notice that the accordian mode natural frequency (larger frequency) of the coupled mass system is nearly the same as the driving frequency. Hmm. This means we should get a large response if we drive at that frequency. So it seems to make sense.

By forcing $a = 0$ which leads to no oscillation in the first mass, the second amplitude $b < 0$ is fixed to oppose the direction of the forcing function (minus sign). The Hookes law force on the first mass exerted by the second spring exactly opposes the force applied to the first mass: $k2\, x_2(t) = -5\cos(10\, t)$. This allows the first mass to remain stationary.

We can explore how the system behaves a range of masses about this value.

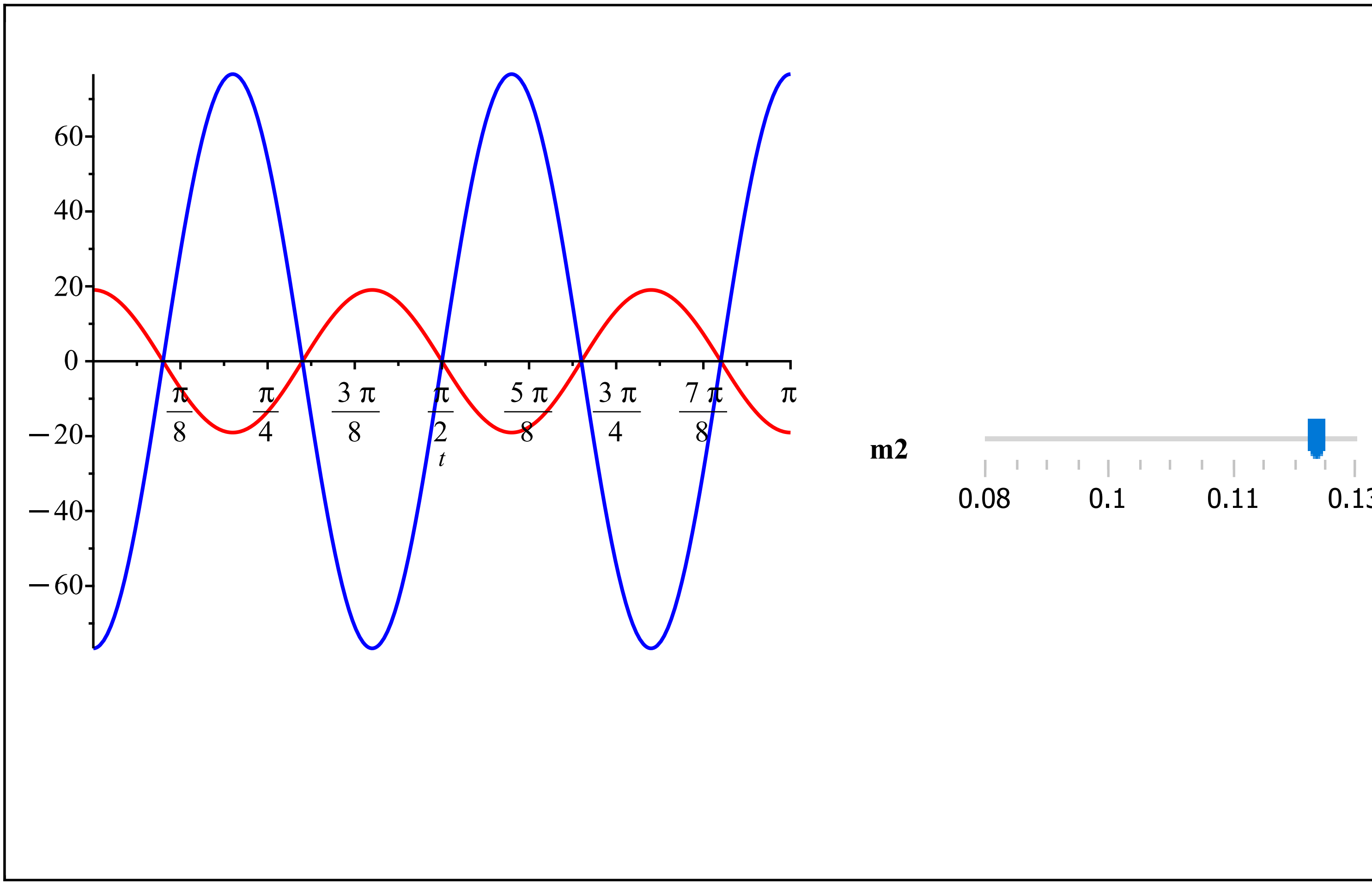
> $solvector := subs(solparams, [a\cos(10\,t), b\cos(10\,t)])$

$$solvector := \left[-\frac{(10\,m2-1)\cos(5\,t)}{10\,(-1+8\,m2)}, \frac{\cos(5\,t)}{10\,(-1+8\,m2)}\right] \quad \textbf{(2.14)}$$

> $\frac{1}{8.}$

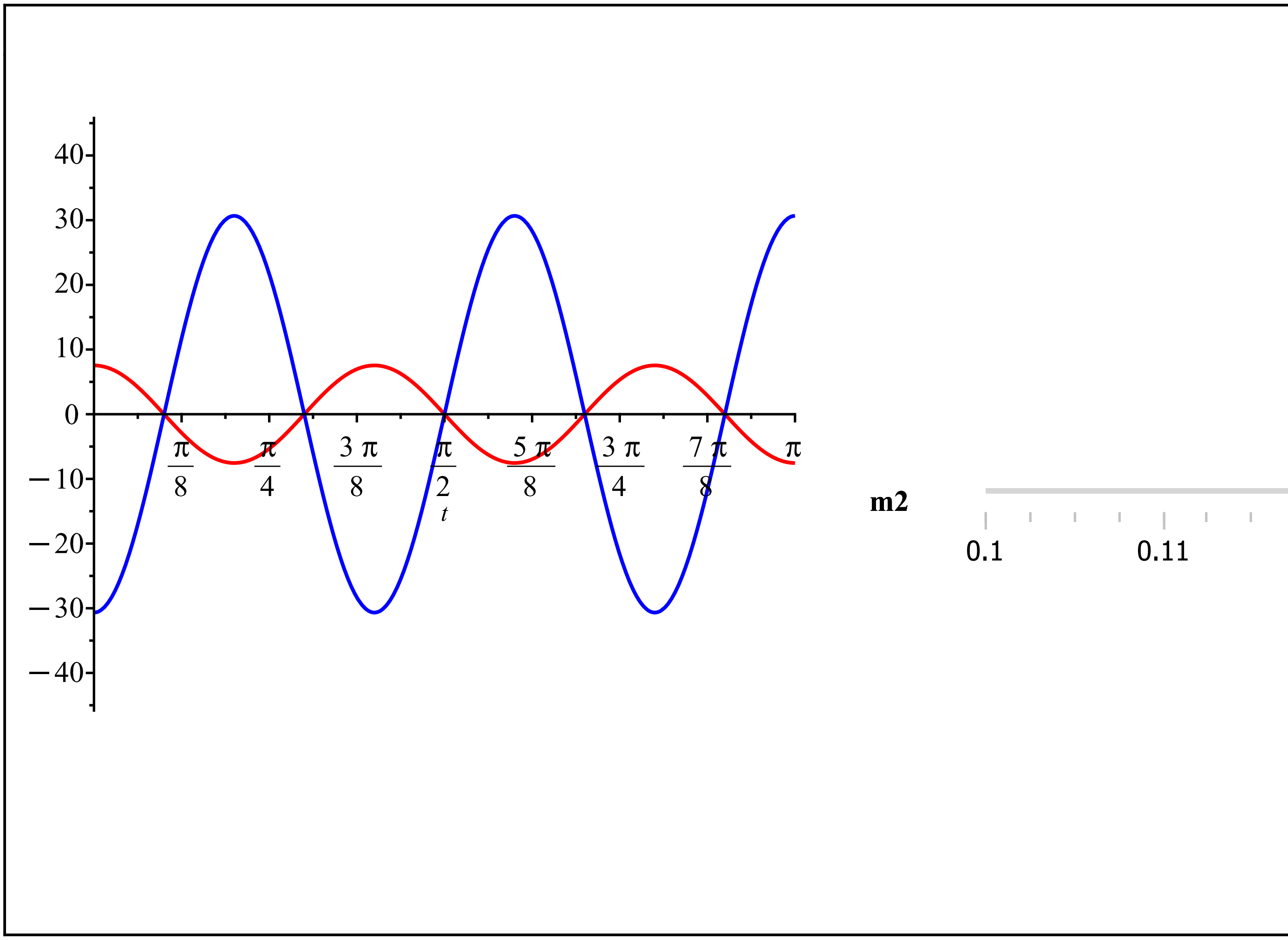
$$0.1250000000 \quad \textbf{(2.15)}$$

> $Explore(plot(solvector, t = 0 .. \pi, color = [red, blue]), m2 = 0.08 ..0.13, placement = right, initialvalues = [m2 = 0.1])$



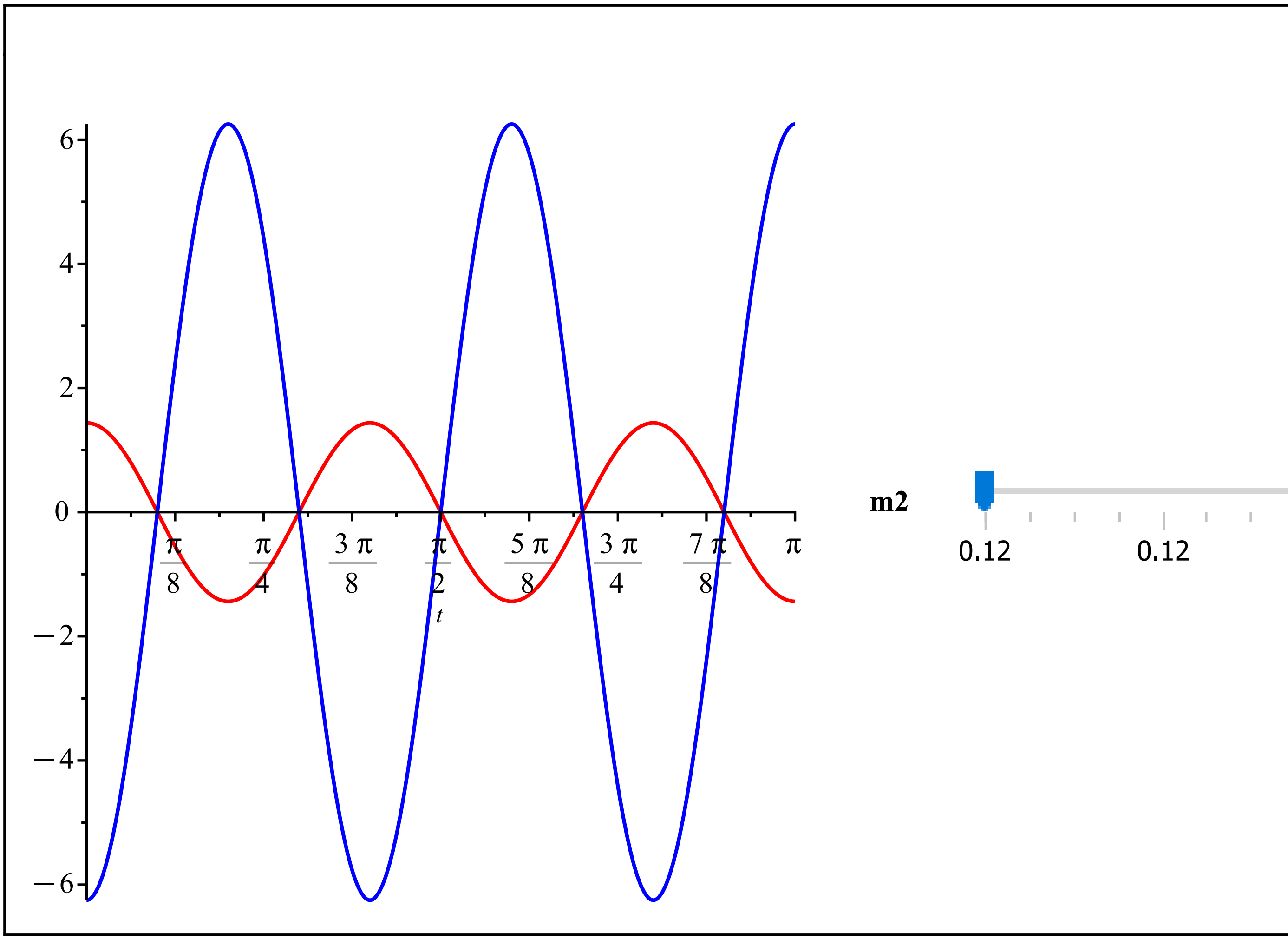
The tandem mode switches to the accordian mode at this special mass value $m = 0.1$, leaving the first mass displacement at zero there as the system switches from tandem to accordian response mode (from both displacements in the same direction to in opposing directions). If we get close to the resonant frequency mass at $m2 = \frac{1}{8} = 0.125$ we see the growth of the amplitudes.

> $Explore(plot(solvector, t = 0 .. \pi, color = [red, blue]), m2 = 0.1 ..0.124, placement = right, initialvalues = [m2 = 0.124])$



Notice at 0.125 how the vertical axis tickmarks grow and then fall sharply.

```
> Explore(plot(solvector, t = 0 .. π, color = [red, blue]), m2 = 0.123 ..0.125, placement = right,
     initialvalues = [m2 = 0.123])
```



movie

We can compare the horizontal oscillations of the steady state solution visually as a function of the second small mass.

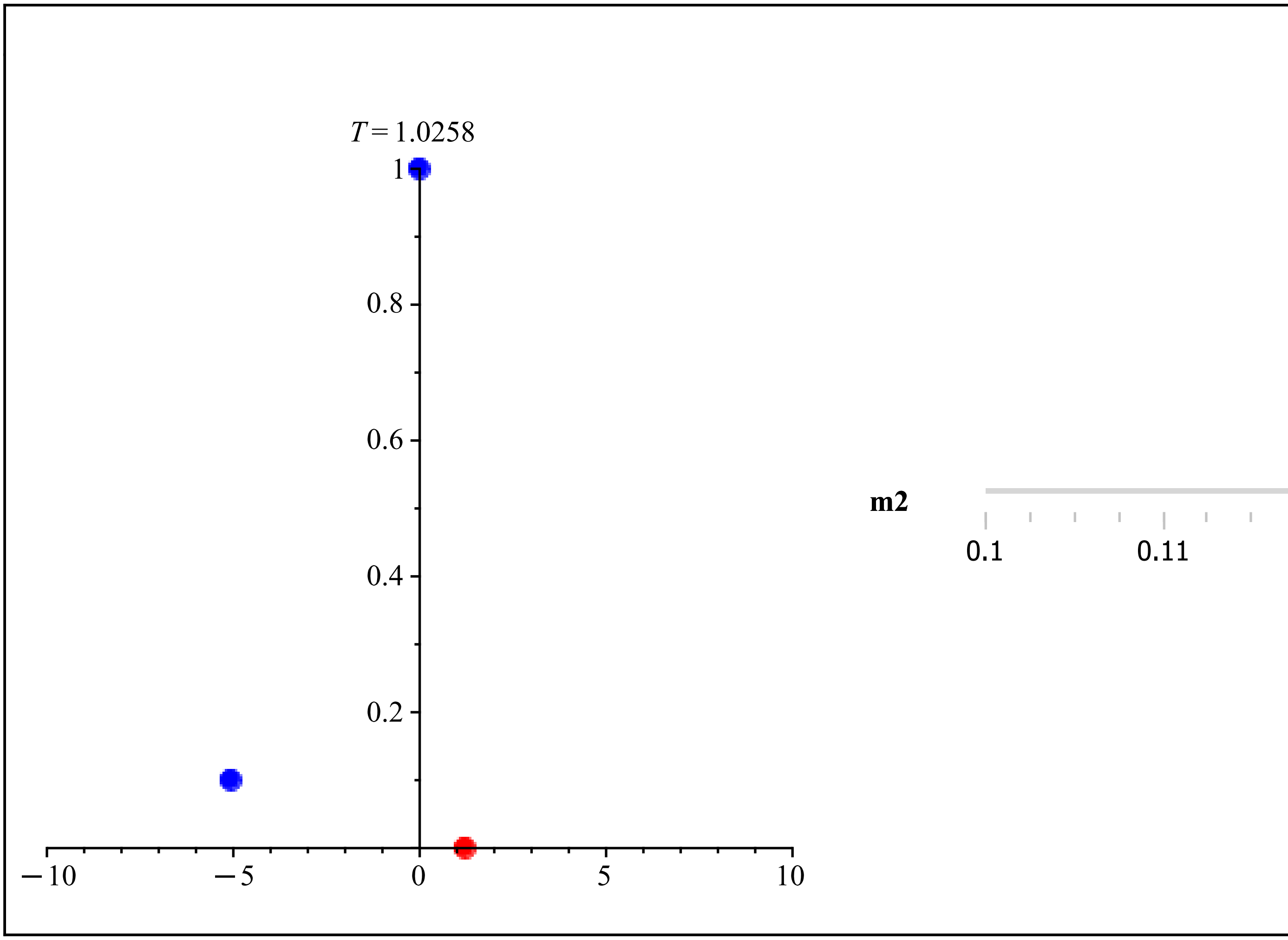
> $with(plots):$

> $Response(m2) := \left[-\frac{(10\, m2 - 1)\cos(5\, t)}{10\,(8\, m2 - 1)}, \frac{\cos(5\, t)}{10\,(8\, m2 - 1)} \right]$

$$Response := m2 \mapsto \left[-\frac{(10 \cdot m2 - 1) \cdot \cos(5 \cdot t)}{80 \cdot m2 - 10}, \frac{\cos(5 \cdot t)}{80 \cdot m2 - 10} \right] \quad \textbf{(3.1)}$$

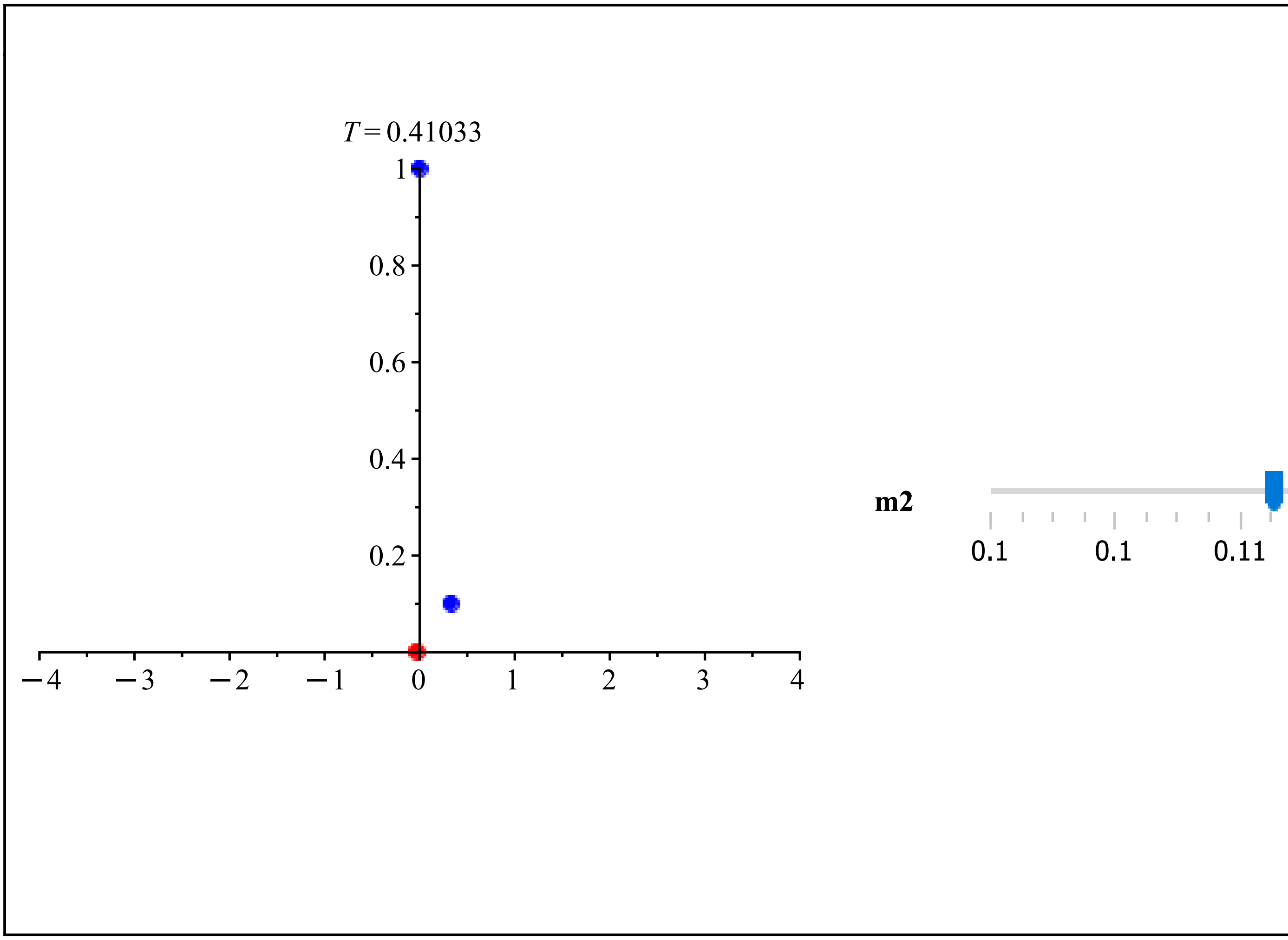
These animations are just to compare the motions side by side for the two masses, with no motion for the first mass at the starting mass value. This is illustrated as a function of the mass in the next section.

> $Explore\Big(animate\Big($
$pointplot, [[[0, 1], [subs(t = T, Response(m2)[1]), 0], [subs(t = T, Response(m2)[2]), 0.1]],$
$symbol = solidcircle, symbolsize = 20, color = [blue, red, blue],$
$view = [-10..10, 0..1]], T = 0..2\frac{\pi}{5}, frames = 50\Big), m2 = 0.1..0.124, initialvalues = [m2 = 0.124],$
$placement = right\Big)$



Very little relative motion near the left endpoint value.

$$> \ Explore\Big(animate\Big(pointplot, [[[0, 1], [subs(t=T, Response(m2)[1]), 0], [subs(t=T, Response(m2)[2]), 0.1]], symbol=solidcircle, symbolsize=20, color=[blue, red, blue], view=[-4..4, 0..1]], T=0..2\frac{\pi}{5}, frames=50\Big), m2=0.1..0.11, initialvalues=[m2=0.124], placement=right\Big)$$



resonance at this mass value

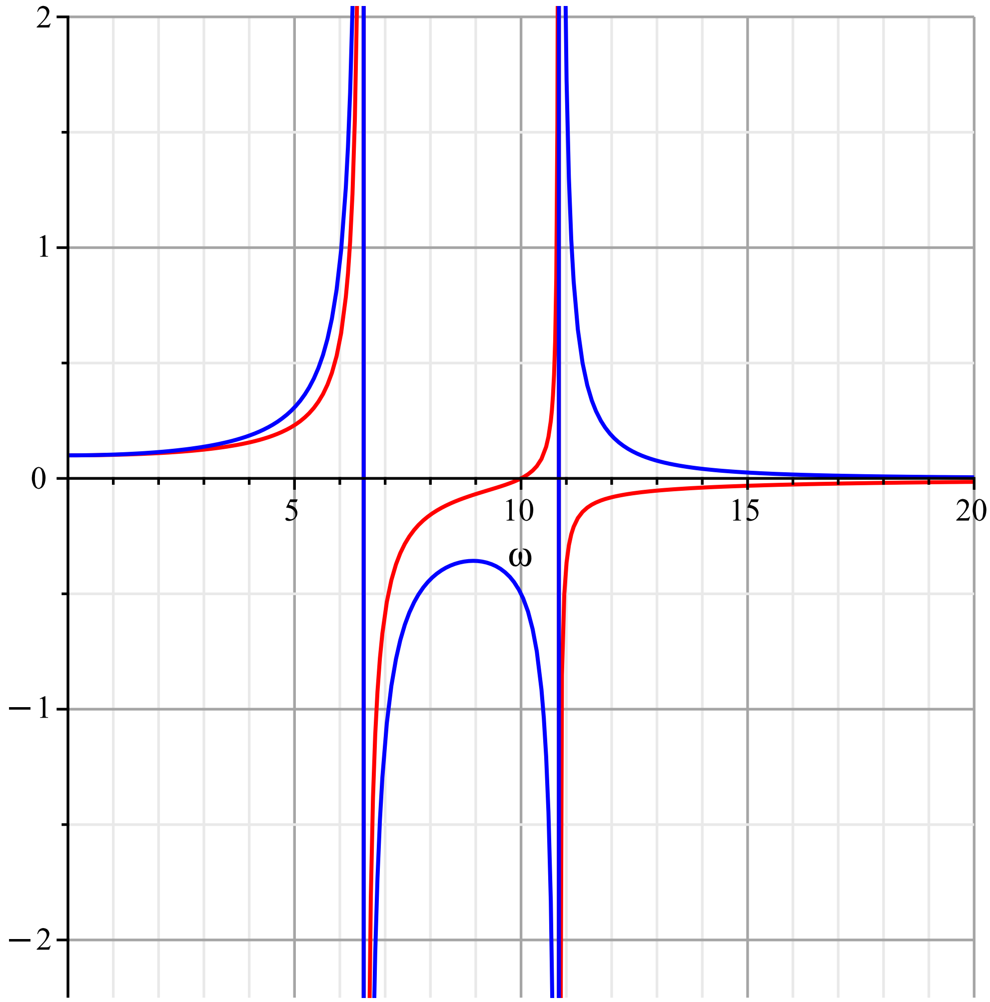
For the chosen mass value we can explore the frequency response to see why the frequency ω = 10 is special for that mass.

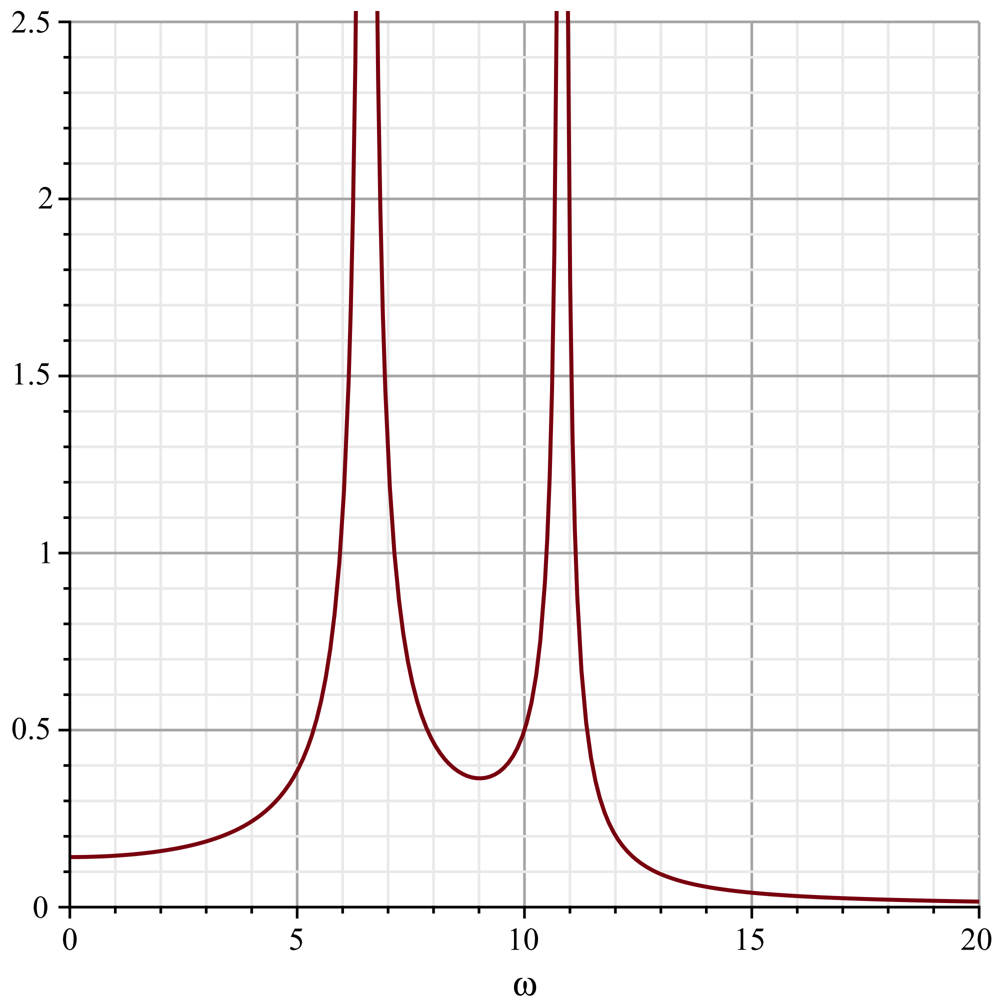
> $deqs3 := \frac{d^2}{dt^2}(x1(t)) + 60\,x1(t) - 10\,x2(t) = 5\cos(\omega\,t),\ \frac{d^2}{dt^2}(x2(t)) - 100\,x1(t) + 100\,x2(t) = 0$

$$deqs3 := \frac{d^2}{dt^2}\,x1(t) + 60\,x1(t) - 10\,x2(t) = 5\cos(\omega\,t),\ \frac{d^2}{dt^2}\,x2(t) - 100\,x1(t) + 100\,x2(t) = 0 \qquad \textbf{(4.1)}$$

> $dsolve(\{deqs3\}, \{x1(t), x2(t)\}) : sol := subs(\_C1 = 0, \_C2 = 0, \_C3 = 0, \_C4 = 0, \%) :$
> $X := subs(sol, [x1(t), x2(t)]) : Amps := simplify\left(\left[\frac{X[1]}{\cos(\omega\,t)}, \frac{X[2]}{\cos(\omega\,t)}\right]\right) :$
> $Amptotal := \sqrt{Amps[1]^2 + Amps[2]^2} :$

> $plot(Amps, \omega = 0..20, gridlines = true, color = [red, blue]); plot(Amptotal, \omega = 0..20, gridlines = true)$


2
1
0
−1
−2
5
10
15
20
ω



Resonance at the two natural frequencies occurs. At low frequency the displacements are both positive in the slow tandem mode, so in phase with the driving force which is the case at zero frequency when the displacements simply go to new equilibrium positions. For very small frequencies, the two masses and springs can keep up with the slowly varying driving force since they can keep up with it, but soon they are both excited by the first natural tandem resonance mode. As we learned for a single mass spring system, when you cross the resonance frequency with weak damping, the phase shift of the response compared to the driving function quickly rises from a small value passing through 90 degrees and then quickly reaching 180 degrees when the response is 180 degrees out of phase with the driving function.

At frequency 10, the crossover to the accordian mode takes place. For these two mass two spring systems there is always such a crossover frequency between the tandem and accordian modes, and so one can tune that crossover point by adjusting some parameter in the system.

Passing through the second resonance again the response switches sign reflecting the usual rapid change in the phase shift by 180 degrees across any resonance peak. As the frequency increases past the second resonance frequency, the system is too sluggish to keep up and the amplitudes both go to zero.

damping? [needs more interpretation]

We add weak damping equally to the two variable DEs:

> $deqs4 := \frac{d^2}{dt^2}(x1(t)) + 60\,x1(t) - 10\,x2(t) - 0.1\,x1'(t) = 5\cos(\omega t),$

$$\frac{d^2}{dt^2}(x2(t)) - 100\,x1(t) + 100\,x2(t) - 0.1\,x2'(t) = 0$$

$$deqs4 := \frac{d^2}{dt^2} x1(t) + 60\, x1(t) - 10\, x2(t) - 0.1\, D(x1)(t) = 5\cos(\omega t), \frac{d^2}{dt^2} x2(t) - 100\, x1(t) + 100\, x2(t) - 0.1\, D(x2)(t) = 0 \quad \textbf{(5.1)}$$

The response functions are not so bad, only one screen approximately.

```
> dsolve({deqs4}, {x1(t), x2(t)}) : sol := subs(_C1 = 0, _C2 = 0, _C3 = 0, _C4 = 0, %) :
  X := subs(sol, [x1(t), x2(t)]);
  Amptotal := sqrt(Amps[1]^2 + Amps[2]^2) :
```

$$X := \Big[-\big(-500000\cos(\omega t)\,\omega^6 - 50000\sin(\omega t)\,\omega^5 + 129995000\cos(\omega t)\,\omega^4 + 9999500\,\omega^3\sin(\omega t) - 10499700000\,\omega^2\cos(\omega t) + 250000000000\cos(\omega t) - 550000000\,\omega\sin(\omega t)\big) \Big/ \Big(10\left(\sqrt{31999 + 4000\sqrt{14}}\,\omega + 10\,\omega^2 + 100\sqrt{14} + 800\right)\left(-\sqrt{31999 + 4000\sqrt{14}}\,\omega + 10\,\omega^2 + 100\sqrt{14} + 800\right)\left(\sqrt{31999 - 4000\sqrt{14}}\,\omega + 10\,\omega^2 - 100\sqrt{14} + 800\right)\left(\sqrt{31999 - 4000\sqrt{14}}\,\omega - 10\,\omega^2 + 100\sqrt{14} - 800\right)\Big), - \big(50000\left(100\,\omega^4 - 16001\,\omega^2 + 500000\right)\cos(\omega t)\big) \Big/ \Big(\left(\sqrt{31999 + 4000\sqrt{14}}\,\omega + 10\,\omega^2 + 100\sqrt{14} + 800\right)\left(-\sqrt{31999 + 4000\sqrt{14}}\,\omega + 10\,\omega^2 + 100\sqrt{14} + 800\right)\left(\sqrt{31999 - 4000\sqrt{14}}\,\omega + 10\,\omega^2 - 100\sqrt{14} + 800\right)\left(\sqrt{31999 - 4000\sqrt{14}}\,\omega - 10\,\omega^2 + 100\sqrt{14} - 800\right)\Big) - \big(1000000\,\omega\left(\omega^2 - 80\right)\sin(\omega t)\big) \Big/ \Big(\left(\sqrt{31999 + 4000\sqrt{14}}\,\omega + 10\,\omega^2 + 100\sqrt{14} + 800\right)\left(-\sqrt{31999 + 4000\sqrt{14}}\,\omega + 10\,\omega^2 + 100\sqrt{14} + 800\right)\left(\sqrt{31999 - 4000\sqrt{14}}\,\omega + 10\,\omega^2 - 100\sqrt{14} + 800\right)\left(\sqrt{31999 - 4000\sqrt{14}}\,\omega - 10\,\omega^2 + 100\sqrt{14} - 800\right)\Big) \Big] \quad \textbf{(5.2)}$$

The damping causes the response modes to be out of phase with the driving force, so we can calculate the phase shift and amplitudes of each variable.

```
> X1c := coeff(X[1], cos(ω t)); X1s := coeff(X[1], sin(ω t));
  X2c := coeff(X[2], cos(ω t)); X2s := coeff(X[2], sin(ω t))
```

$$X1c := -\left(-500000\,\omega^6 + 129995000\,\omega^4 - 10499700000\,\omega^2 + 250000000000\right) \Big/ \Big(10\left(\sqrt{31999 + 4000\sqrt{14}}\,\omega + 10\,\omega^2 + 100\sqrt{14} + 800\right)\left(-\sqrt{31999 + 4000\sqrt{14}}\,\omega + 10\,\omega^2 + 100\sqrt{14} + 800\right)\left(\sqrt{31999 - 4000\sqrt{14}}\,\omega + 10\,\omega^2 - 100\sqrt{14}\right.$$

$$+ 800\big) \left(\sqrt{31999 - 4000\sqrt{14}}\ \omega - 10\,\omega^2 + 100\sqrt{14} - 800\right)\Big)$$

$$X1s := -\left(-50000\,\omega^5 + 9999500\,\omega^3 - 550000000\,\omega\right) \Big/ \Big(10\,\Big(\sqrt{31999 + 4000\sqrt{14}}\ \omega + 10\,\omega^2 + 100\sqrt{14} + 800\Big)\Big(-\sqrt{31999 + 4000\sqrt{14}}\ \omega + 10\,\omega^2 + 100\sqrt{14} + 800\Big)\Big(\sqrt{31999 - 4000\sqrt{14}}\ \omega + 10\,\omega^2 - 100\sqrt{14} + 800\Big)\Big(\sqrt{31999 - 4000\sqrt{14}}\ \omega - 10\,\omega^2 + 100\sqrt{14} - 800\Big)\Big)$$

$$X2c := -\left(50000\left(100\,\omega^4 - 16001\,\omega^2 + 500000\right)\right) \Big/ \Big(\Big(\sqrt{31999 + 4000\sqrt{14}}\ \omega + 10\,\omega^2 + 100\sqrt{14} + 800\Big)\Big(-\sqrt{31999 + 4000\sqrt{14}}\ \omega + 10\,\omega^2 + 100\sqrt{14} + 800\Big)\Big(\sqrt{31999 - 4000\sqrt{14}}\ \omega + 10\,\omega^2 - 100\sqrt{14} + 800\Big)\Big(\sqrt{31999 - 4000\sqrt{14}}\ \omega - 10\,\omega^2 + 100\sqrt{14} - 800\Big)\Big)$$

$$X2s := -\left(1000000\,\omega\left(\omega^2 - 80\right)\right) \Big/ \Big(\Big(\sqrt{31999 + 4000\sqrt{14}}\ \omega + 10\,\omega^2 + 100\sqrt{14} + 800\Big)\Big(-\sqrt{31999 + 4000\sqrt{14}}\ \omega + 10\,\omega^2 + 100\sqrt{14} + 800\Big)\Big(\sqrt{31999 - 4000\sqrt{14}}\ \omega + 10\,\omega^2 - 100\sqrt{14} + 800\Big)\Big(\sqrt{31999 - 4000\sqrt{14}}\ \omega - 10\,\omega^2 + 100\sqrt{14} - 800\Big)\Big) \quad \textbf{(5.3)}$$

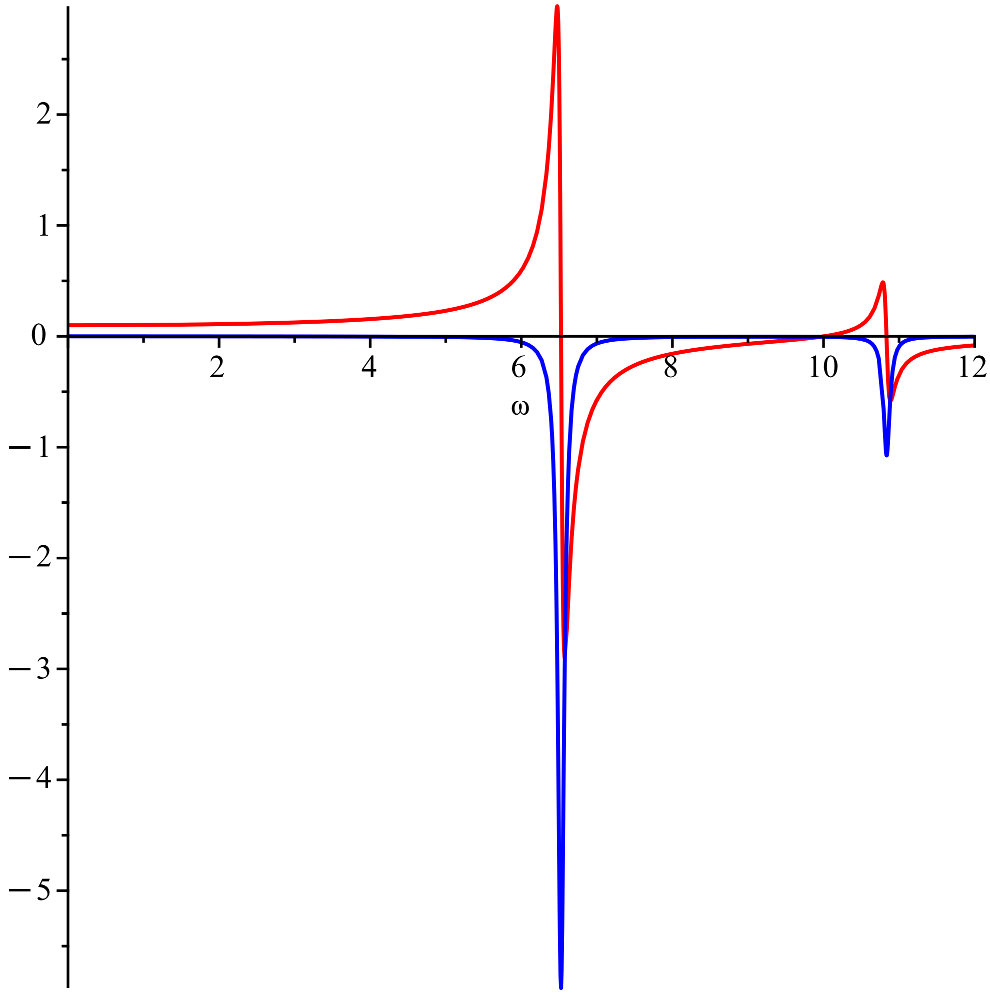
Take a peak at the small frequency values of the coefficients of the cosine and sine of the two response vector components. [What does this mean?]

```
> eval( [X1c, X1s], ω = 0.1 ); evalf( eval( [X2c, X2s], ω = 0.1 ) )
```

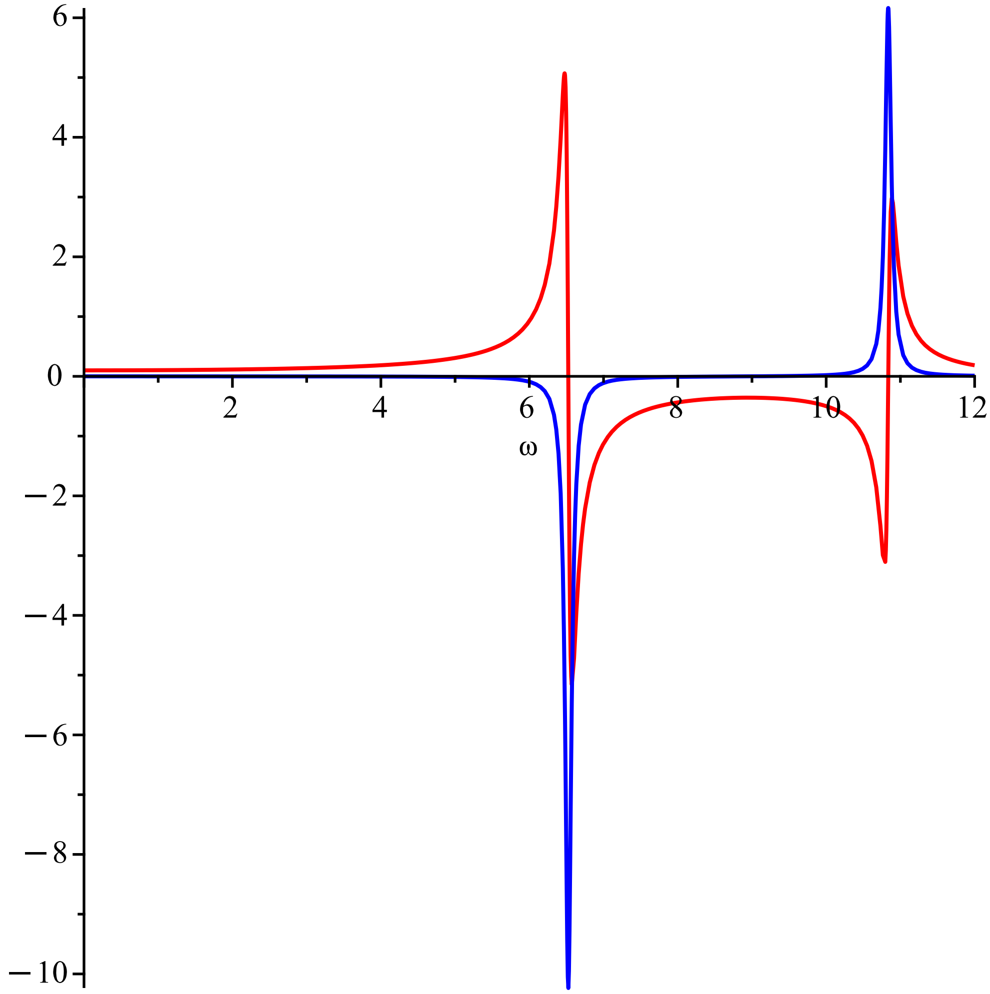
$$\Big[-\left(2.498950160 \times 10^{10}\right) \Big/ \Big(\Big(0.1\sqrt{31999 + 4000\sqrt{14}} + 800.10 + 100\sqrt{14}\Big)\Big(-0.1\sqrt{31999 + 4000\sqrt{14}} + 800.10 + 100\sqrt{14}\Big)\Big(0.1\sqrt{31999 - 4000\sqrt{14}} + 800.10 - 100\sqrt{14}\Big)\Big(0.1\sqrt{31999 - 4000\sqrt{14}} - 800.10 + 100\sqrt{14}\Big)\Big), \left(5.499000100 \times 10^{6}\right) \Big/ \Big(\Big(0.1\sqrt{31999 + 4000\sqrt{14}} + 800.10 + 100\sqrt{14}\Big)\Big(-0.1\sqrt{31999 + 4000\sqrt{14}} + 800.10 + 100\sqrt{14}\Big)\Big(0.1\sqrt{31999 - 4000\sqrt{14}} + 800.10 - 100\sqrt{14}\Big)\Big(0.1\sqrt{31999 - 4000\sqrt{14}} - 800.10 + 100\sqrt{14}\Big)\Big)\Big]$$

$$[0.1000320000,\ -0.00003201648398] \quad \textbf{(5.4)}$$

```
> plot( [X1c, X1s], ω = 0 ..12, color = [red, blue] ); plot( [X2c, X2s], ω = 0 ..12, color = [red, blue] )
```



2
1
0
−1
−2
−3
−4
−5
2
4
6
8
10
12
ω



[What does the next sentence mean?]

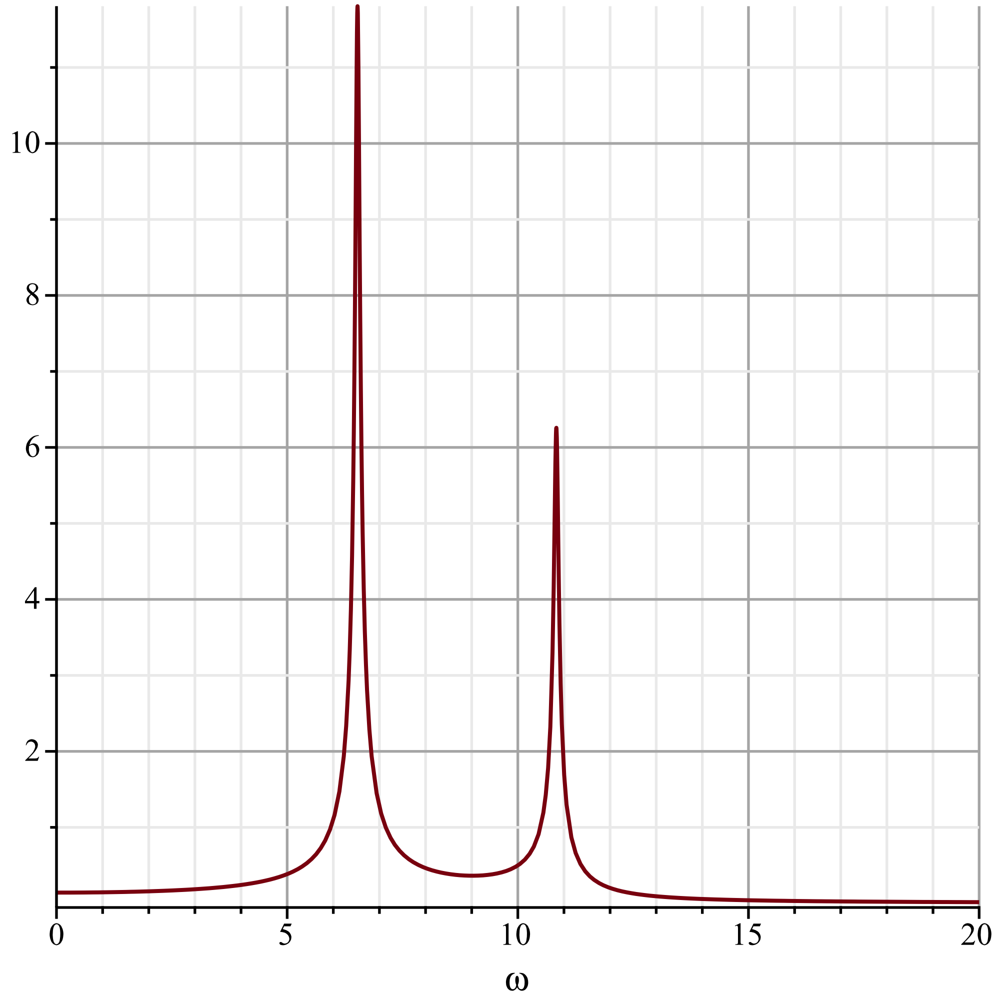
Looking at the plot below, as the frequency increases from zero, the sine coefficient goes negative so the phase shift in the sinusoidal coefficient plane goes into the fourth quadrant where the response functions lead the driving force, which is unexpected. I would have thought the phase shift would go positive to reflect a lagging behind behavior. See the plot below.

First we plot the separate amplitudes of each variable and then the total amplitude of the response vector function. The switch in sign now goes into the phase shift changing by 180 degrees, also plotted below.

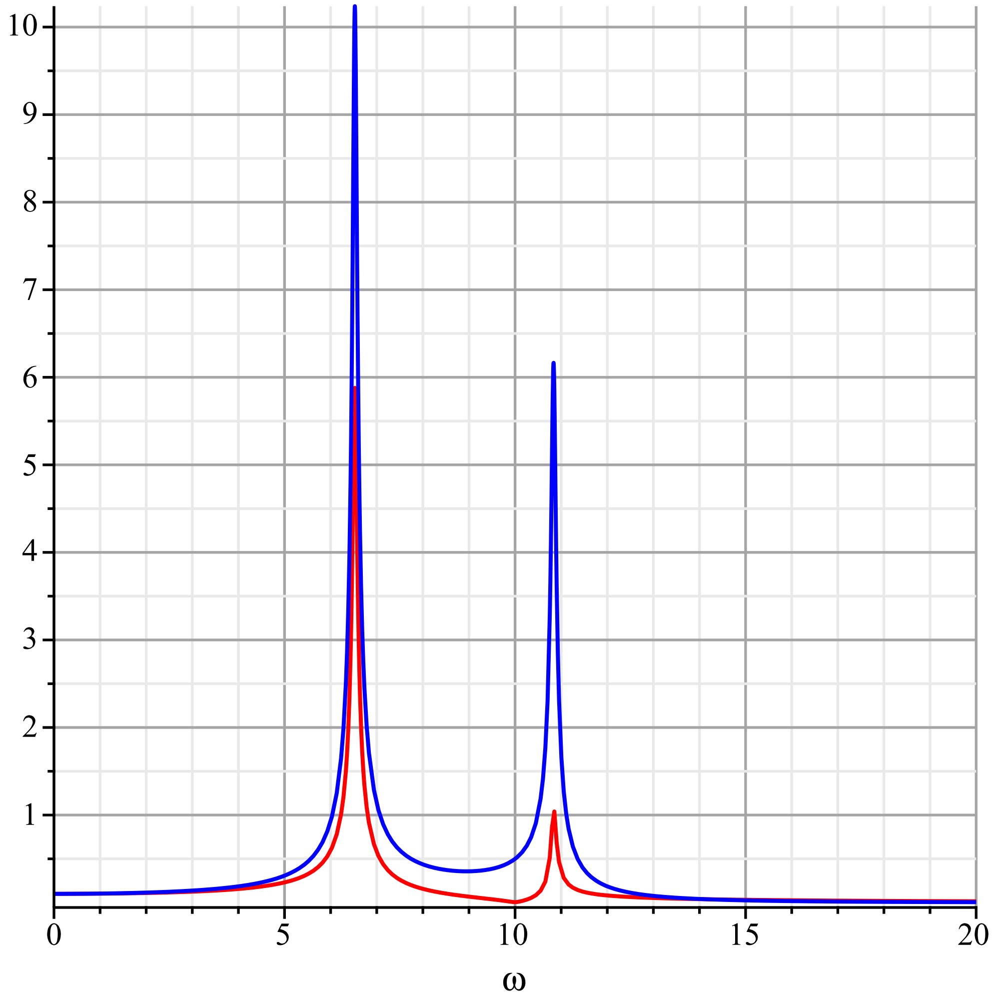
> $X1amp := \sqrt{X1c^2 + X1s^2} : X2amp := \sqrt{X2c^2 + X2s^2} : AmpTotal := \sqrt{X1amp^2 + X2amp^2} :$

> $X1delta := \arctan(X1s, X1c) : X2delta := \arctan(X1s, X1c) :$

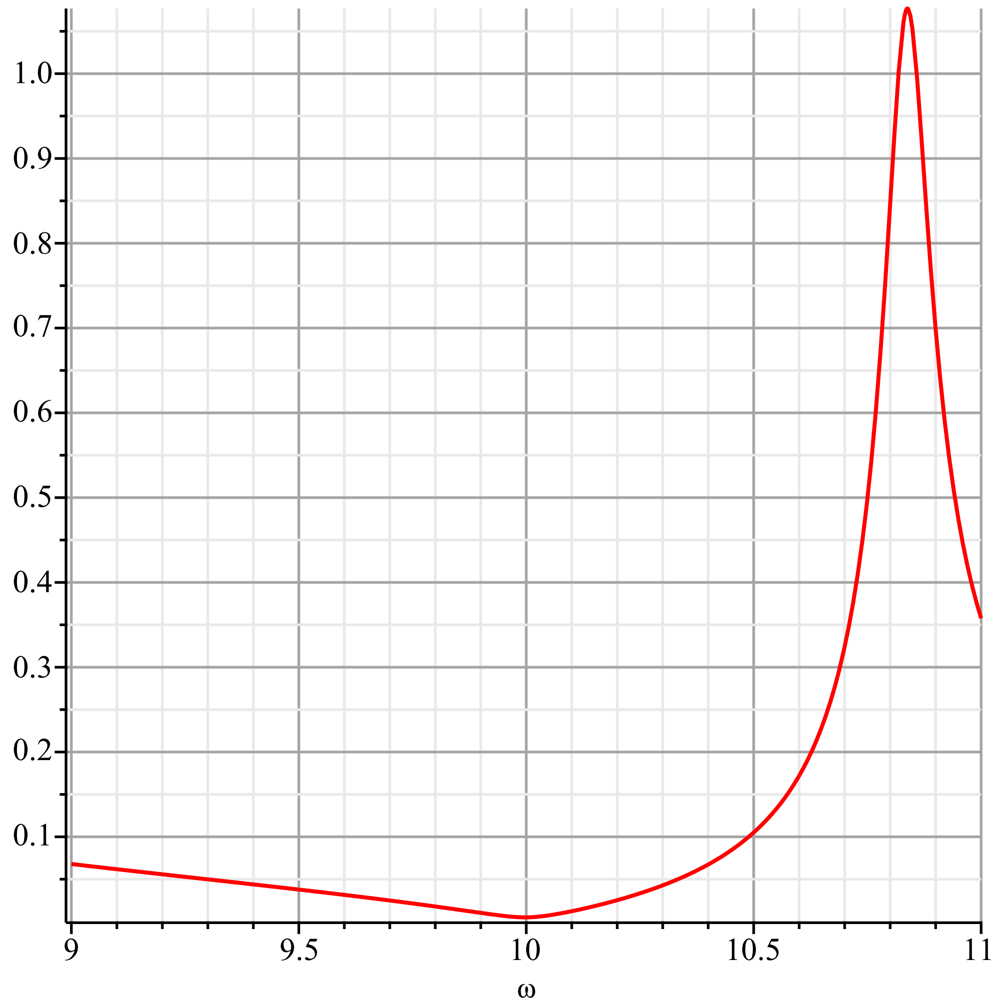
> $plot(AmpTotal, \omega = 0\,..20, gridlines = true)$



> $plot([X1amp, X2amp], \omega = 0..20, color = [red, blue], gridlines = true)$



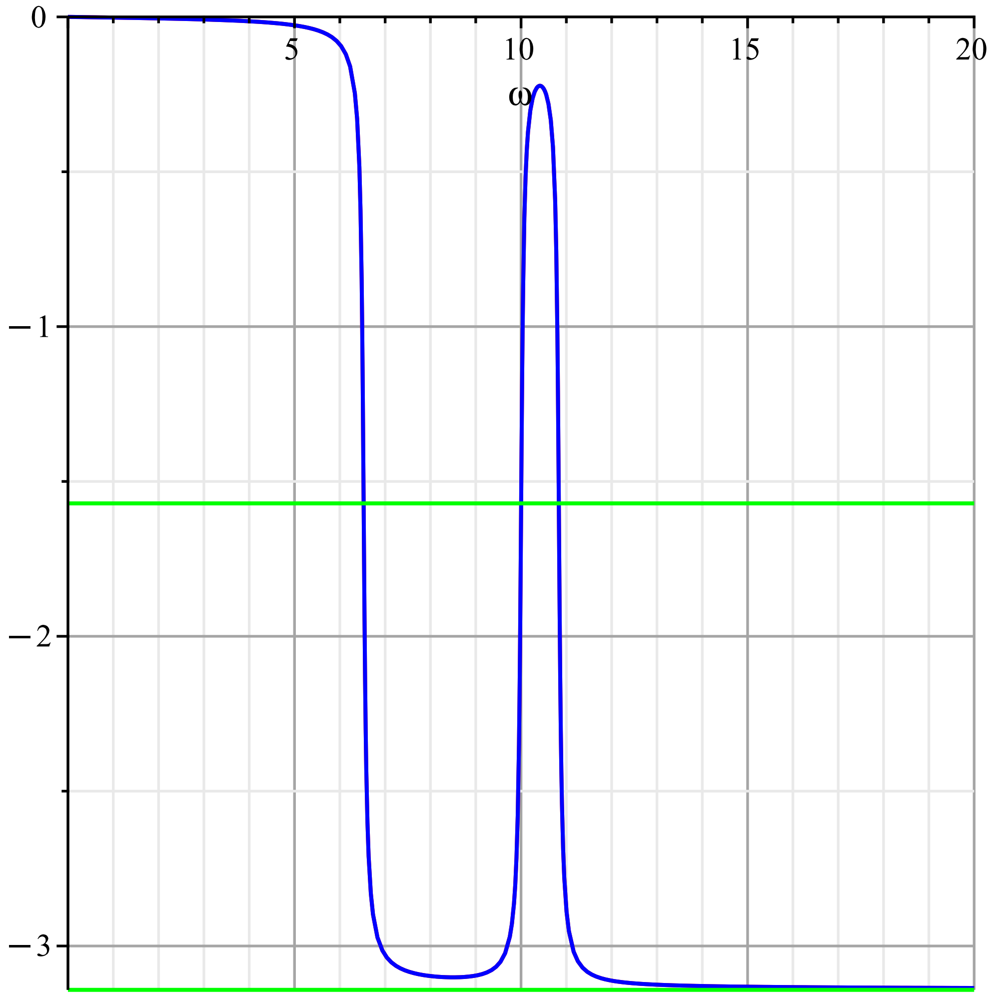
> *plot*([*X1amp*], ω = 9 ..11, *color* = [*red*, *blue*], *gridlines* = *true*); *subs*(ω = 10., *X1amp*)



0.004991021741 **(5.5)**

Very small but not zero.

> $plot\left(\left[X1delta, X2delta, -\frac{\pi}{2}, -\pi\right], \omega = 0\,..20, color = [red, blue, green\$2], gridlines = true\right)$



This is curious, the phase shift goes in the opposite direction compared to the single mass spring system, through negative values leading the driving force instead of lagging behind, but indeed passes through negative 90 degrees quickly to 180 degrees passing through the first resonance frequency. It then briefly returns towards zero as it passes through the second resonance and then returns to 180 degrees. I do not understand this. It requires more investigation.

parameter analysis

> *with*(*LinearAlgebra*) :

> $A := \begin{bmatrix} -\left(\omega_1^2 + \zeta\,\omega_2^2\right) & \zeta\,\omega_2^2 \\ \omega_2^2 & -\omega_2^2 \end{bmatrix}$

$$A := \begin{bmatrix} -\zeta\,\omega_2^2 - \omega_1^2 & \zeta\,\omega_2^2 \\ \omega_2^2 & -\omega_2^2 \end{bmatrix} \quad \textbf{(6.1)}$$

> *LinearAlgebra*:-*Eigenvectors*(**(6.1)**)

(6.2)

$$\left[\begin{array}{c} -\frac{\zeta\omega_2^2}{2}-\frac{\omega_1^2}{2}-\frac{\omega_2^2}{2}+\frac{\sqrt{\zeta^2\omega_2^4+2\zeta\omega_1^2\omega_2^2+2\omega_2^4\zeta+\omega_1^4-2\omega_1^2\omega_2^2 \cdots}}{2} \\ -\frac{\zeta\omega_2^2}{2}-\frac{\omega_1^2}{2}-\frac{\omega_2^2}{2}-\frac{\sqrt{\zeta^2\omega_2^4+2\zeta\omega_1^2\omega_2^2+2\omega_2^4\zeta+\omega_1^4-2\omega_1^2\omega_2^2 \cdots}}{2} \end{array}\right], \quad (6.2)$$

$$\left[\begin{array}{c} \frac{\zeta\omega_2^2}{\frac{\zeta\omega_2^2}{2}+\frac{\omega_1^2}{2}-\frac{\omega_2^2}{2}+\frac{\sqrt{\zeta^2\omega_2^4+2\zeta\omega_1^2\omega_2^2+2\omega_2^4\zeta+\omega_1^4-2\omega_1^2\omega_2^2+\cdots}}{2}} \cdots \\ 1 \quad \cdots \end{array}\right]$$

> $Den := \zeta^2\omega_2^4+2\zeta\omega_1^2\omega_2^2+2\omega_2^4\zeta+\omega_1^4-2\omega_1^2\omega_2^2+\omega_2^4;\ simplify\left(Den-\left(\omega_1^2-\zeta\omega_2^2\right)^2\right)$

$$Den := \zeta^2\omega_2^4+2\zeta\omega_1^2\omega_2^2+2\zeta\omega_2^4+\omega_1^4-2\omega_1^2\omega_2^2+\omega_2^4$$

$$(2\zeta+1)\,\omega_2^4+4\,\omega_1^2\left(\zeta-\frac{1}{2}\right)\omega_2^2 \quad (6.3)$$

We can make it a sum of squares except for one term, not sure if this observation can help further analysis.

> $Den=\left(\omega_1^2-\zeta\omega_2^2\right)^2+\omega_2^2\left((2\zeta+1)\,\omega_2^2+4\,\omega_1^2\left(\zeta-\frac{1}{2}\right)\right);\ simplify(lhs(\%)-rhs(\%))$

$$\zeta^2\omega_2^4+2\zeta\omega_1^2\omega_2^2+2\zeta\omega_2^4+\omega_1^4-2\omega_1^2\omega_2^2+\omega_2^4=\left(-\zeta\omega_2^2+\omega_1^2\right)^2+\omega_2^2\left((2\zeta+1)\,\omega_2^2+4\,\omega_1^2\left(\zeta-\frac{1}{2}\right)\right)$$

$$0 \quad (6.4)$$

The DE system is $x''=Ax+F$, or $Ax-x''=F$ so if $F=\cos(\omega t)\begin{bmatrix}1\\0\end{bmatrix}$ for a unit driving force on the first mass, and the response is $x=\cos(\omega t)\begin{bmatrix}a(\omega)\\b(\omega)\end{bmatrix}$ we get $Aomega\,x=F$ or explicitly the coefficient matrix

> $Aomega := A+\omega^2\, IdentityMatrix(2)$

$$Aomega := \begin{bmatrix} -\zeta\omega_2^2+\omega^2-\omega_1^2 & \zeta\omega_2^2 \\ \omega_2^2 & \omega^2-\omega_2^2 \end{bmatrix} \quad (6.5)$$

> $\begin{bmatrix}a\\b\end{bmatrix}=Aomega^{-1}\begin{bmatrix}1\\0\end{bmatrix}$

(6.6)

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -\dfrac{\omega^2 - \omega_2^2}{\zeta\,\omega^2\,\omega_2^2 - \omega^4 + \omega^2\,\omega_1^2 + \omega^2\,\omega_2^2 - \omega_1^2\,\omega_2^2} \\ \dfrac{\omega_2^2}{\zeta\,\omega^2\,\omega_2^2 - \omega^4 + \omega^2\,\omega_1^2 + \omega^2\,\omega_2^2 - \omega_1^2\,\omega_2^2} \end{bmatrix} \tag{6.6}$$

>

So the first mass will remain fixed in the response mode if the natural frequency of the uncoupled second mass spring system is the driving frequency. This enables the natural oscillation of the decoupled second mass spring system to exactly balance the force applied to the first mass, and hence leave it fixed.