$\boxed{7.3b}$ ①

$x_1' = x_1 + 2x_2$

$x_2' = -4x_1 - 3x_2$

$x_1(0) = 1$

$x_2(0) = 2$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \overbrace{\begin{bmatrix} 1 & 2 \\ -4 & -3 \end{bmatrix}}^{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

complex eigenvalue case

(a bit sloppy! sorry)

$$0 = |A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ -4 & -3-\lambda \end{vmatrix} = (1-\lambda)(-1)(3+\lambda) + 8 = (\lambda+3)(\lambda-1) + 8$$
$$= \lambda^2 + 2\lambda - 3 + 8 = \lambda^2 + 2\lambda + 5$$

$$\lambda = \frac{-2 \pm \sqrt{4 - 4(+5)}}{2} = -1 \pm \sqrt{1-5} = -1 \pm 2i$$

$$A - (-1+2i)I = \begin{bmatrix} 1+1-2i & 2 \\ -4 & -3+1-2i \end{bmatrix} = \begin{bmatrix} 2-2i & 2 \\ -4 & -2-2i \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & +\frac{1+i}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

F

$x_2 = t$, $x_1 = -\frac{1+i}{2} t$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \underbrace{\begin{bmatrix} -\frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}}_{b_1}$$

$$e^{\lambda_1} b_1 = e^{(-1+2i)t} \begin{bmatrix} -\frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix} = e^{-t} (\cos 2t + i \sin 2t) \begin{bmatrix} -\frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

$$= e^{-t} \left[\left(-\tfrac{1}{2} \cos 2t + \tfrac{1}{2} \sin 2t\right) + i \left(-\tfrac{1}{2} \sin 2t - \tfrac{1}{2} \cos 2t\right) \right]$$

$$\equiv X_1 + i X_2$$

$\{X_1 \pm i X_2\}$ = basis of soln space

↳ switch to real basis $\{X_1, X_2\}$

gen soln:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 X_1 + c_2 X_2 = e^{-t} \left(c_1 \begin{bmatrix} -\frac{1}{2} \cos 2t + \frac{1}{2} \sin 2t \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -\frac{1}{2} \sin 2t - \frac{1}{2} \cos 2t \\ \sin 2t \end{bmatrix} \right)$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} c_1 - \frac{1}{2} c_2 \\ c_1 \end{bmatrix}$$

$\rightarrow c_1 = 2$

$\rightarrow 1 = -\cancel{\tfrac{1}{2}(2)} - \frac{1}{2} c_2$

$c_2 = 4$

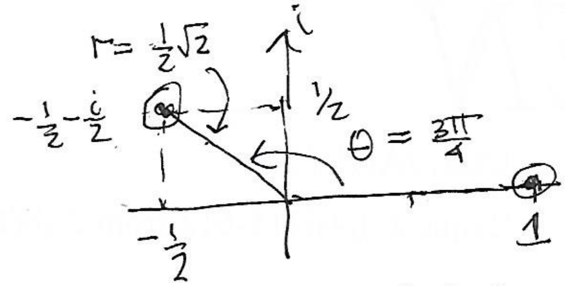
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = e^{-t} \left(2 \begin{bmatrix} -\frac{1}{2} \cos 2t + \frac{1}{2} \sin 2t \\ \cos 2t \end{bmatrix} - 4 \begin{bmatrix} -\frac{1}{2} \sin 2t - \frac{1}{2} \cos 2t \\ \sin 2t \end{bmatrix} \right)$$

$$= e^{-t} \begin{bmatrix} -\cos 2t + \sin 2t + 2 \sin 2t + 2 \cos 2t \\ 2 \cos 2t - 4 \sin 2t \end{bmatrix}$$

$$= e^{-t} \begin{bmatrix} \cos 2t + 3 \sin 2t \\ 2 \cos 2t - 4 \sin 2t \end{bmatrix} = e^{-t} \begin{bmatrix} \sqrt{10} \cos(2t - \arctan 3) \\ 2\sqrt{5} \cos(2t + \arctan 2) \end{bmatrix} = e^{-t} \begin{bmatrix} A_1 \cos(2t - \delta_1) \\ A_2 \cos(2t - \delta_2) \end{bmatrix}$$

$\boxed{7.3b}$ 2

$$b_1 = \begin{bmatrix} -\frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} e^{\frac{3\pi}{4} i} \\ 1\, e^{0i} \end{bmatrix}$$

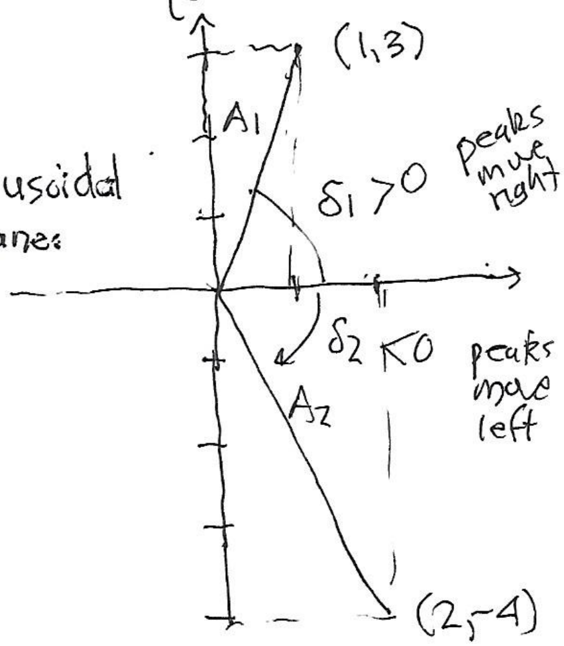


complex plane

← relative amplitude (ratio) and relative phase shift (difference) of two oscillations fixed by complex eigenvector

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = e^{-t} \begin{bmatrix} \sqrt{10} \cos(2t - \arctan 3) \\ 2\sqrt{5} \cos(2t + \arctan 2) \end{bmatrix} = e^{-t} \begin{bmatrix} \cos 2t + 3 \sin 2t \\ 2\cos 2t - 4 \sin 2t \end{bmatrix}$$

sinusoidal plane:



$\frac{A_2}{A_1} = \frac{2\sqrt{5}}{\sqrt{10}} = \sqrt{2}$ x_2 bigger than x_1 by this factor

$\delta_2 - \delta_1 = \arctan 2 + \arctan 3 \underset{\text{Maple}}{=} \frac{3\pi}{4}$

$\frac{3\pi/4}{2\pi} = \frac{3}{8} = 0.375$

about 1/3 cycle.

x_2 is to the left of x_1 by about 1/3 cycle (ahead in time)

(see Maple)