7.4:3-9 2mass 3 spring resonance exercise summary

$$\vec{x}'' = A\vec{x} + \vec{F} \qquad \vec{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix},\ \underbrace{A = \begin{bmatrix}-3&2\\1&-2\end{bmatrix}}_{\lambda = -1,-4},\ \vec{F} = \begin{bmatrix}0\\60\cos\omega t\end{bmatrix}$$

$$\begin{pmatrix}\vec{x} = B\vec{y}\\ \vec{y} = B^{-1}\vec{x}\end{pmatrix}\downarrow \qquad B = \begin{bmatrix}1&-2\\1&1\end{bmatrix},\ B^{-1} = \frac{1}{3}\begin{bmatrix}1&2\\-1&1\end{bmatrix},\ A_B = B^{-1}AB = \begin{bmatrix}-1&0\\0&-4\end{bmatrix}$$

$$B^{-1}(B\vec{y})'' = B^{-1}(AB\vec{x} + \vec{F}) \qquad B^{-1}\vec{F} = \frac{60\cos\omega t}{3}\begin{bmatrix}1&2\\-1&1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}40\cos\omega t\\20\cos\omega t\end{bmatrix}$$

$$\vec{y}'' = A_B\vec{y} + B^{-1}\vec{F}$$

$$\begin{bmatrix}y_1''\\y_2''\end{bmatrix} = \begin{bmatrix}-1&0\\0&-4\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix} + \begin{bmatrix}40\cos\omega t\\20\cos\omega t\end{bmatrix} = \begin{bmatrix}-y_1 + 40\cos\omega t\\-4y_2 + 20\cos\omega t\end{bmatrix}$$

$$\boxed{\begin{aligned}y_1'' + y_1 &= 40\cos\omega t\\ y_2'' + 4y_2 &= 20\cos\omega t\end{aligned}}$$

decoupled equations — look for particular soln via method of undetermined constants → "response functions"
[homogeneous soln involves 4 arbitrary coefficients later set by initial conditions]

$y_1'' + y_1 = 40\cos\omega t$

$1\ [y_{1p} = c_5\cos\omega t + c_6\sin\omega t]$
$0\ [y_{1p}' = -c_5\omega\sin\omega t + c_6\omega\cos\omega t]$
$1\ [y_{1p}'' = -c_5\omega^2\cos\omega t - c_6\omega^2\sin\omega t]$

$$y_{1p}'' + y_{1p} = \underbrace{(1-\omega^2)c_5}_{=40}\cos\omega t + \underbrace{(1-\omega^2)c_6}_{=0}\sin\omega t = 40\cos\omega t$$

$c_5 = \dfrac{40}{1-\omega^2}$ $\qquad c_6 = 0$

$y_2'' + 4y_2 = 20\cos\omega t$

$4\ (y_{2p} = C_7\cos\omega t + C_8\sin\omega t)$
$0\ [y_{2p}' = -C_7\omega\sin\omega t + C_8\omega\cos\omega t]$
$1\ [y_{2p}'' = -C_7\omega^2\cos\omega t - C_8\omega^2\sin\omega t]$

$$y_{2p}'' + 4y_{2p} = \underbrace{(4-\omega^2)C_7}_{=20}\cos\omega t + \underbrace{(4-\omega^2)C_8}_{=0}\sin\omega t = 20\cos\omega t$$

$C_7 = \dfrac{20}{4-\omega^2}$ $\qquad C_8 = 0$

since no damping terms in system

$$\vec{y}_p = \begin{bmatrix}y_{1p}\\y_{2p}\end{bmatrix} = \begin{bmatrix}40/(1-\omega^2)\ \cos\omega t\\20/(4-\omega^2)\ \cos\omega t\end{bmatrix}$$

$$\vec{x}_p = B\vec{y}_p = \begin{bmatrix}1&-2\\1&1\end{bmatrix}\begin{bmatrix}40/(1-\omega^2)\\20/(4-\omega^2)\end{bmatrix}\cos\omega t = 20\cos\omega t\begin{bmatrix}\frac{2}{1-\omega^2} - \frac{2}{4-\omega^2}\\ \frac{2}{1-\omega^2} + \frac{1}{4-\omega^2}\end{bmatrix} = \frac{20\cos\omega t}{(1-\omega^2)(4-\omega^2)}\begin{bmatrix}2(4-\omega^2) - 2(1-\omega^2)\\2(4-\omega^2) + (1-\omega^2)\end{bmatrix}$$

$$= \frac{20\cos\omega t}{(1-\omega^2)(4-\omega^2)}\begin{bmatrix}6\\9-3\omega^2\end{bmatrix} = \vec{x}_p(0)\cos\omega t,$$

response function oscillates between tips of vectors $\vec{x}_p(0), -\vec{x}_p(0)$ back & forth

$$\vec{x}_p(0) = \frac{60}{\underbrace{(1-\omega^2)(4-\omega^2)}}\begin{bmatrix}2\\3-\omega^2\end{bmatrix} \xrightarrow{\omega=3} \frac{60}{(-8)(-5)}\begin{bmatrix}2\\-6\end{bmatrix} = \begin{bmatrix}3\\-9\end{bmatrix}$$

previous result ✓

$\omega \neq 1, 2$ natural frequencies characterizing homogeneous soln
very big response near these frequencies — resonance at them:
solution at $\omega = 1$ or $\omega = 2$ leads to oscillation amplitudes which are proportional to t, thus growing without bound.

Notice at $\omega = \sqrt{3}$, $x_{2p}(t) = 0$ second variable constant while first variable alone oscillates