E&P2 7.3.9

$$x_1' = 2x_1 - 5x_2 \quad x_1(0)=2 \qquad \longleftrightarrow \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \underbrace{\begin{bmatrix} 2 & -5 \\ 4 & -2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$
$$x_2' = 4x_1 - 2x_2 \quad x_2(0)=3$$

$$\underline{A} - \lambda \underline{I} = \begin{bmatrix} 2-\lambda & -5 \\ 4 & -2-\lambda \end{bmatrix} \qquad \det(A-\lambda I) = (2-\lambda)(-1)(2+\lambda) + 20 = \lambda^2 - 4 + 20 = \lambda^2 + 16, \quad \lambda = \pm 4i$$

$$\lambda = 4i: \quad \underline{A - 4iI} = \begin{bmatrix} 2-4i & -5 \\ 4 & -2-4i \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} \overset{L}{1} & \overset{F}{-\tfrac{1}{2}(1+2i)} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{matrix} x_2 = t \\ x_1 = \tfrac{1}{2}(1+2i)t \end{matrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (\tfrac{1}{2}+i)t \\ t \end{bmatrix} = t \underbrace{\begin{bmatrix} \tfrac{1}{2}+i \\ 1 \end{bmatrix}}_{E_1} \qquad \lambda = -4i, \quad E_2 = \overline{E}_1 = \begin{bmatrix} \tfrac{1}{2} - i \\ 1 \end{bmatrix}$$

$$B = [E_1\ \overline{E}_1] = \begin{bmatrix} \tfrac{1}{2}+i & \tfrac{1}{2}-i \\ 1 & 1 \end{bmatrix} \qquad \underline{x} = \underline{B}\,\underline{y}, \quad \underline{y} = \underline{B}^{-1}\underline{x}, \quad \underline{A_B} = \underline{B}^{-1}\underline{A}\,\underline{B} = \begin{bmatrix} 4i & 0 \\ 0 & -4i \end{bmatrix}$$

$$\underline{x}' = \underline{A}\,\underline{x} \rightarrow \underline{y}' = \underline{A_B}\,\underline{y} \qquad \begin{matrix} y_1' = 4i\, y_1 \\ y_2' = -4i\, y_2 \end{matrix} \qquad \begin{matrix} y_1 = c_1 e^{4it} \\ y_2 = c_2 e^{-4it} \end{matrix}$$

$$\underline{x} = \underline{B}\,\underline{y} = \begin{bmatrix} \tfrac{1}{2}+i & \tfrac{1}{2}-i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{4it} \\ c_2 e^{-4it} \end{bmatrix} = c_1 \underbrace{e^{4it} \begin{bmatrix} \tfrac{1}{2}+i \\ 1 \end{bmatrix}} + c_2 \underbrace{e^{-4it} \begin{bmatrix} \tfrac{1}{2}-i \\ 1 \end{bmatrix}}$$

complex basis of soln space.

real & imaginary parts of these vector solns are a real basis of the soln space:

$$e^{4it} \begin{bmatrix} \tfrac{1}{2}+i \\ 1 \end{bmatrix} = (\cos 4t + i \sin 4t) \begin{bmatrix} \tfrac{1}{2}+i \\ 1 \end{bmatrix} = \begin{bmatrix} \tfrac{1}{2}\cos 4t + \tfrac{i}{2}\sin 4t + i\cos 4t - \sin 4t \\ \cos 4t + i \sin 4t \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \tfrac{1}{2}\cos 4t - \sin 4t \\ \cos 4t \end{bmatrix}} + i \underbrace{\begin{bmatrix} \tfrac{1}{2}\sin 4t + \cos 4t \\ \sin 4t \end{bmatrix}}$$

new real basis

(complex conjugate has same real part, opp. signed imag part)

gen soln:
$$\underline{x} = c_1 \begin{bmatrix} \tfrac{1}{2}\cos 4t - \sin 4t \\ \cos 4t \end{bmatrix} + c_2 \begin{bmatrix} \tfrac{1}{2}\sin 4t + \cos 4t \\ \sin 4t \end{bmatrix} = \begin{bmatrix} (\tfrac{1}{2}c_1 + c_2)\cos 4t + (-c_1 + \tfrac{1}{2}c_2)\sin 4t \\ c_1 \cos 4t + c_2 \sin 4t \end{bmatrix}$$

$$\underline{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \tfrac{1}{2}c_1 + c_2 \\ c_1 \end{bmatrix} = \begin{bmatrix} \tfrac{1}{2} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \tfrac{1}{2} & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = -1 \begin{bmatrix} 0 & -1 \\ -1 & \tfrac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1/2 \end{bmatrix}$$

$$-c_1 + \tfrac{1}{2}c_2 = -3 + \tfrac{1}{4} = -11/4$$

ivp soln:
$$\underline{x} = \begin{bmatrix} 2\cos 4t - 11/4 \sin 4t \\ 3 \cos 4t + 1/2 \sin 4t \end{bmatrix}$$

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interpretation
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{8^2+11^2}}{4} \cos(4t + \arctan(11/8)) \\ \frac{\sqrt{6^2+1^2}}{2} \cos(4t - \arctan(1/6)) \end{bmatrix} \approx \begin{bmatrix} 3.400 \cos(4t + 0.9420) \\ 3.041 \cos(4t - 0.1651) \end{bmatrix} \leftarrow \text{slightly larger amplitude}$$

x_1 has peaks ahead of x_2 by $0.9420 + 0.1651$ rad $= 63.4°$ or $\Delta t = \frac{1.1071}{4} = .2768$
(in time) $= 1.1071$ rad