1st order linear homogeneous DE system: purely imaginary eigenvalues

$x_1' = 4x_2$
$x_2' = -4x_1$
$x_1(0) = 1$
$x_2(0) = 0$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 4 \\ -4 & -\lambda \end{vmatrix} = \lambda^2 + 16 = 0 \rightarrow \lambda = \pm 4i$$

$$\lambda = 4i: \begin{bmatrix} -4i & 4 \\ -4 & -4i \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} \overset{L}{1} & \overset{F}{i} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad x_1 = -ix_2 = -it, \quad x_2 = t$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -it \\ t \end{bmatrix} = t \underbrace{\begin{bmatrix} -i \\ 1 \end{bmatrix}}_{\vec{b}_1} \rightarrow \vec{b}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$\underline{B} = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \quad \underline{x} = \underline{B}\underline{y},\ \underline{y} = \underline{B}^{-1}\underline{x},\ \underline{A}_B = \begin{bmatrix} 4i & 0 \\ 0 & -4i \end{bmatrix}$$

$$\underline{x}' = \underline{A}\underline{x} \rightarrow \underline{y}' = \underline{A}_B\underline{y}: \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 4i & 0 \\ 0 & -4i \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \begin{matrix} y_1' = 4iy_1 \\ y_2' = -4iy_2 \end{matrix} \quad \begin{matrix} y_1 = C_1 e^{4it} \\ y_2 = C_2 e^{-4it} \end{matrix}$$

$$\underline{x} = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C_1 e^{4it} \\ C_2 e^{-4it} \end{bmatrix} = C_1 e^{4it} \begin{bmatrix} -i \\ 1 \end{bmatrix} + C_2 e^{-4it} \begin{bmatrix} i \\ 1 \end{bmatrix} \longrightarrow \text{complex conjugate basis solutions}$$

find Re, Im parts of either one

$$e^{4it} \begin{bmatrix} -i \\ 1 \end{bmatrix} = (\cos 4t + i \sin 4t) \begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} \sin 4t - i\cos 4t \\ \cos 4t + i \sin 4t \end{bmatrix} = \underbrace{\begin{bmatrix} \sin 4t \\ \cos 4t \end{bmatrix}}_{\underline{B}_1(t)} + i \underbrace{\begin{bmatrix} -\cos 4t \\ \sin 4t \end{bmatrix}}_{\underline{B}_2(t)}$$

real basis of soln space

$$\rightarrow \underline{x} = c_1 \underline{B}_1(t) + c_2 \underline{B}_2(t) = c_1 \begin{bmatrix} \sin 4t \\ \cos 4t \end{bmatrix} + c_2 \begin{bmatrix} -\cos 4t \\ \sin 4t \end{bmatrix} \quad \text{general solution}$$

$$= \begin{bmatrix} c_1 \sin 4t - c_2 \cos 4t \\ c_1 \cos 4t + c_2 \sin 4t \end{bmatrix} \longleftarrow \text{final form, can read off } x_1(t), x_2(t)$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -c_2 \\ c_1 \end{bmatrix} \rightarrow \begin{matrix} c_2 = -1 \\ c_1 = 0 \end{matrix}$$

$$\underline{x} = \begin{bmatrix} \cos 4t \\ -\sin 4t \end{bmatrix} \quad \text{IVP solution (just } -\underline{B}_2(t)\text{!)}$$

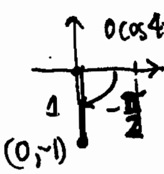
$$= \begin{bmatrix} \cos 4t \\ \cos(4t + \pi/2) \end{bmatrix}$$

same amplitude

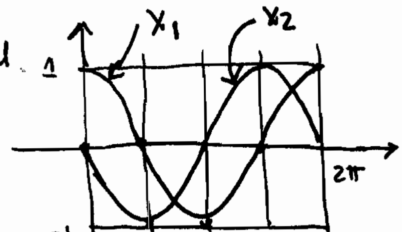
x_2 is 90° ahead of x_1

ahead in time = earlier time → shifted left

$0\cos 4t - \sin 4t = \cos(4t + \pi/2)$



WHAT YOU SEE



same

polar form of complex numbers makes complex multiplication easy:

$$\underline{x} = 2\mathrm{Re}\left\{ C_1 e^{4it} \begin{bmatrix} -i \\ 1 \end{bmatrix} \right\} = 2\mathrm{Re}\left\{ \left(\tfrac{A_0}{2} e^{i\delta_0}\right) e^{4it} \begin{bmatrix} 1e^{-i\pi/2} \\ 1e^{i0} \end{bmatrix} \right\}$$

equal amplitudes

$$= \mathrm{Re}\begin{bmatrix} A_0 e^{i(4t + \delta_0 - \pi/2)} \\ A_0 e^{i(4t + \delta_0)} \end{bmatrix} = \begin{bmatrix} A_0 \cos(4t + \delta_0 - \pi/2) \\ A_0 \cos(4t + \delta_0) \end{bmatrix} \longleftarrow x_1 \text{ is } 90° \text{ behind } x_2$$

behind in time = later time → peaks shifted right = "ahead on graph"

overall amplitude and initial phase set by initial conditions,

relative amplitudes, relative phase set by DE system, recorded in polar form of eigenvectors

aside:

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = B^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} i/2 \\ -i/2 \end{bmatrix}$$

$$2C_1 = i = e^{i\pi/2}$$

$$\underline{x} = \mathrm{Re}\begin{bmatrix} e^{i4t} \\ e^{i(4t + \pi/2)} \end{bmatrix}$$