Edwards and Penney, 2nd Ed, 5.6.23

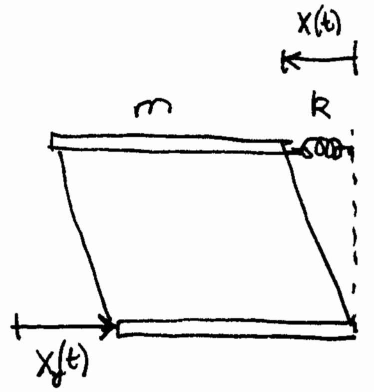
A building consists of 2 floors.
The 1st floor is rigidly attached to the ground and the 2nd floor is of mass 1000 slugs (fps units) and weighs 16 tons (32,000 lbs). (← "mg" never needed)

The elastic frame of the building behaves as a spring that resists horizontal displacements of the second floor; it requires a horizontal force of 5 tons to displace the second floor a distance of 1 ft.

Assume that in an earthquake the ground oscillates horizontally with amplitude A_0 and circular frequency ω, resulting in an external horizontal force $F(t) = m A_0 \omega^2 \sin\omega t$ on the second floor.

a) what is the natural frequency (Hz) of oscillations of the second floor?

b) If the ground undergoes one oscillation every 2.25 s with an amplitude of 3 in, what is the amplitude of the resulting forced oscillations of the second floor?



FL2 displaced from equilibrium by $X(t)$

FL1 moves

$X_0(t) = A_0 \sin\omega t \rightarrow X_0''(t) = -A_0\omega^2 \sin\omega t$

acceleration → force:

$m X_0''(t) = -mA_0\omega^2 \sin\omega t$

on floor 2, but opposite to stretch so

$F(t) = mA_0\omega^2 \sin\omega t$

driving frequency:

$\omega = \dfrac{2\pi}{2.25}\,\dfrac{\text{rad}}{\text{sec}} = 2.79\,\dfrac{\text{rad}}{\text{s}}\,\dfrac{\text{cycle}}{2\pi\,\text{rad}}$

$= 0.44\ \text{Hz} \neq \omega_0$ but close

driving amplitude:

$A_0 = 3\,\text{in} = 0.25\,\text{ft}$

a) Use Hooke's Law to determine spring constant

$F = kx$

$5\,\text{tons} = k(1\,\text{ft})$

$5(2000\,\text{lbs})$

$\therefore\ k = 10{,}000\ \text{lbs/ft}$

Next: equation of motion

$$\underbrace{m x''}_{\text{mass} \times \text{acc.}} = \underbrace{-kx}_{\text{restoring force}} + \underbrace{m A_0\omega^2 \sin\omega t}_{\text{applied force to whole building}}$$

or $$x'' + \left(\frac{k}{m}\right)x = A_0\omega^2 \sin\omega t$$

ω_0^2 (natural frequency) $\rightarrow \omega_0 = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{10000\ \text{lbs/ft}}{1000\ \text{slugs}}}$

$= \sqrt{10}\,\dfrac{\text{rad}}{\text{sec}}$ ← (must be inverse time unit)

$\approx 3.16\ \text{rad/s} \times \left(\dfrac{\text{cycle}}{2\pi\,\text{rad}}\right) = .50\ \text{cycles/s}$

$= .50\ \text{Hz}$

b) $x'' + \omega_0^2 x = A_0\omega^2 \sin\omega t$

$\underbrace{x_h = c_1\cos\omega_0 t + c_2 \sin\omega_0 t}$

ignore this part of motion to do problem
both modes present in total solution

$\omega \neq \omega_0$: $[x_p = c_3\cos\omega t + c_4\sin\omega t]$

$0\,[x_p' = -c_3\omega\sin\omega t + c_4\omega\cos\omega t]$

$1\,[x_p'' = -c_3\omega^2\cos\omega t - c_4\omega^2\sin\omega t]$

$$x_p'' + \omega_0^2 x_p = \underbrace{c_3(\omega_0^2-\omega^2)}_{=0}\cos\omega t + \underbrace{c_4(\omega_0^2-\omega^2)}_{=A_0\omega^2}\sin\omega t = A_0\omega^2\sin\omega t$$

$c_3 = 0,\quad c_4 = \dfrac{A_0\omega^2}{\omega_0^2-\omega^2}$

$x_p = \dfrac{A_0\omega^2}{\omega_0^2-\omega^2}\sin\omega t$

amplitude of 1st floor oscillation

amplitude of 2nd floor oscillation

amplification factor:

$\dfrac{\omega^2}{\omega_0^2-\omega^2} = \dfrac{(2.79)^2}{(3.16)^2-(2.79)^2}$

$= 3.536$

$(3\,\text{in})(3.536) = 10.61\,\text{in}$

not enough significant figures in intermediate numbers — only 3!!

repeat:

$3\left(\dfrac{(2\pi/2.25)^2}{10-(2\pi/2.25)^2}\right)$

$= 10.625\ \text{in}$ ✓

☺