E&P 5.6.17 resonance calculation

$x'' + 6x' + 45x = 50\cos\omega t$ Find the amplitude $A(\omega)$ of the steady-state response function x_p (the particular soln from the method of undetermined constants) and find both the peak value $\omega_p > 0$ for which $A'(\omega_p) = 0$ as well as $A(\omega_p)$ and then compare the ratio $A(\omega_p)/A(0)$ with the quality factor $Q = \omega_0\tau_0$.

$1\,x'' + k_0 x' + \omega_0^2 x = 50\cos\omega t \;\rightarrow\; k_0 = 6 \rightarrow \tau_0 = 1/k_0 = 1/6,\quad \omega_0 = \sqrt{45} = 3\sqrt{5} \approx 6.71$

$Q = \omega_0\tau_0 = \frac{1}{6}3\sqrt{5} = \frac{\sqrt{5}}{2} \approx 1.12$ (slightly underdamped)

<u>Aside:</u> The transient solution is the homogeneous soln of this D.E. only necessary to describe how the system transitions from the initial conditions to the steady state: $r^2 + 6r + 45 = 0 \rightarrow r = -3 \pm 6i \rightarrow x_h = e^{-3t}(c_1\cos 6t + c_2\sin 6t)$

This decays to less than 1% of its initial amplitude by $t = 5/3$.

$45\,[x_p = c_3\cos\omega t + c_4\sin\omega t]$

$6\,[x_p' = -\omega c_3\sin\omega t + \omega c_4\cos\omega t]$

$1\,[x_p'' = -\omega^2 c_3\cos\omega t - \omega^2 c_4\sin\omega t]$

$$x_p'' + 6x_p' + 45x_p = [\underbrace{(45-\omega^2)c_3 + 6\omega c_4}_{=50}]\cos\omega t + [\underbrace{-6\omega c_3 + (45-\omega^2)c_4}_{=0}]\sin\omega t = 50\cos\omega t$$

$$\begin{bmatrix} 45-\omega^2 & 6\omega \\ -6\omega & 45-\omega^2 \end{bmatrix}\begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 50 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \frac{1}{(45-\omega^2)^2 + 36\omega^2}\begin{bmatrix} 45-\omega^2 & -6\omega \\ 6\omega & 45-\omega^2 \end{bmatrix}\begin{bmatrix} 50 \\ 0 \end{bmatrix} = \frac{50}{(45-\omega^2)^2 + 36\omega^2}\begin{bmatrix} 45-\omega^2 \\ 6\omega \end{bmatrix}$$

$$A(\omega) = \sqrt{c_3^2 + c_4^2} = \frac{50}{(45-\omega^2)^2 + 36\omega^2}\sqrt{(45-\omega^2)^2 + 36\omega^2} = \frac{50}{\sqrt{(45-\omega^2)^2 + 36\omega^2}}$$

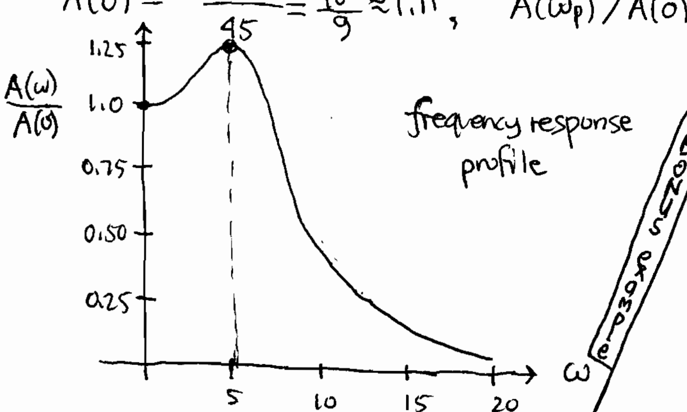
$$0 = A'(\omega) = \frac{d}{d\omega}\, 50\left((45-\omega^2)^2 + 36\omega^2\right)^{-1/2} = -\frac{1}{2}\left((45-\omega^2)^2 + 36\omega^2\right)^{-3/2} 50 \cdot [2(45-\omega^2)(-2\omega) + 36(2\omega)]$$

$\rightarrow -2(45-\omega^2) + 36 = 0 \rightarrow 45 - \omega^2 - 18 = 0 \rightarrow \omega^2 = 27 \rightarrow \boxed{\omega_p = \sqrt{27} = 3\sqrt{3} \approx 5.19 \text{ compared to } \omega_0 = 5\sqrt{3} \approx 6.71}$

$$A(\omega_p) = \frac{50}{\sqrt{(45-27)^2 + 36(27)}} = \frac{50}{\sqrt{18^2 + (2\cdot 18)(9\cdot 3)}} = \frac{50}{\sqrt{18^2(1+3)}} = \frac{50}{2\cdot 18} = \frac{50}{36} = \frac{25}{18} \approx 1.39$$

$A(0) = \frac{50}{45} = \frac{10}{9} \approx 1.11$, $\quad A(\omega_p)/A(0) = \frac{50/36}{50/45} = \frac{45}{36} = \frac{5}{4} = 1.25 \leftrightarrow (Q = 1.12)$

(slightly bigger)



initial conditions at rest at equilibrium: $x(0) = 0 = x'(0)$ at peak frequency ω_p

$x = x_h + x_p = e^{-3t}(c_1\cos 6t + c_2\sin 6t) + c_3\cos\sqrt{27}t + c_4\sin\sqrt{27}t$

$x' = -3e^{-3t}(c_1\cos 6t + c_2\sin 6t) + e^{-3t}(-6c_1\sin 6t + 6c_2\cos 6t)$
$\quad - \sqrt{27}c_3\sin\sqrt{27}t + \sqrt{27}c_4\cos\sqrt{27}t$

$x(0) = c_1 + c_3 = 0 \rightarrow c_1 = -c_3$

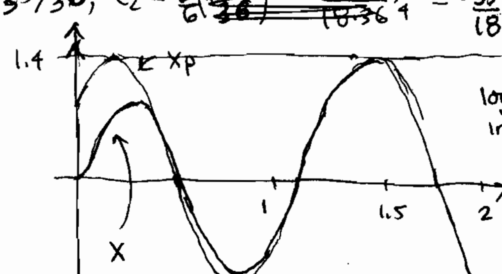
$x'(0) = -3c_1 + 6c_2 + \sqrt{27}c_4 = 0 \rightarrow c_2 = (3c_1 - \sqrt{27}c_4)/6$

help maple!

$c_1 = -35/36$, $c_2 = \frac{3}{6}(\ldots)^3 = -\frac{35}{18}$

$$\begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \frac{50}{18^2 + 32\cdot 27}\begin{bmatrix} 18 \\ \underbrace{6\sqrt{27}}_{=18\sqrt{3}} \end{bmatrix} = \frac{50}{18\cdot 4}\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} = \begin{bmatrix} 25/36 \\ \frac{25\sqrt{3}}{36} \end{bmatrix}$$

$$x = \frac{35}{36}e^{-3t}[-\cos 6t - 2\sin 6t] + \frac{25}{36}[\cos\sqrt{27}t + \sqrt{3}\sin\sqrt{27}t]$$



looks better in Maple!