E&P3 5.5.9 → altered to become a physically interesting DE (driven damped harmonic motion)
(overdamped case)

$$y'' + 3y' + 2y = x e^{-x} \equiv f(x)$$

$r = -1 \rightarrow r + 1 = 0$, $m = 2$

$(D+1)^2 (x e^{-x}) = 0$

$y_p = (c_2 + c_3 x) e^{-x}$

↓ satisfies hom D.E. so multiply by x

$y_p = x(c_2 + c_3 x) e^{-x}$

$= (c_2 x + c_3 x^2) e^{-x}$

$r^2 + 3r + 2 = 0$

$(r+2)(r+1) = 0$

$r = -2, -1$

$e^{rx} = e^{-2x}, e^{-x}$

$y_h = c_1 e^{-2x} + c_2 e^{-x}$

$(D+1)^2 (D+2)(D+1) y = 0$

$(D+2)(D+1)^3 y = 0$

$r = -2 \quad r = -1$, $m = 3$

$$y = c_1 e^{-2x} + (c_2 + c_3 x + c_4 x^2) e^{-x}$$

$$= y_h + \underbrace{(c_3 x + c_4 x^2) e^{-x}}_{y_p}$$

(optional) Exercise: backsubstitute y_p into D.E. to determine c_3, c_4 :

find result: $y_p = \left(\frac{1}{2}x^2 - x\right) e^{-x}$

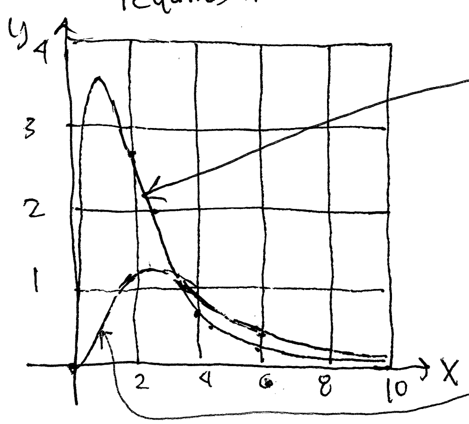
so $y = c_1 e^{-2x} + c_2 e^{-x} + \left(\frac{1}{2}x^2 - x\right) e^{-x}$

Exercise:
Think of x as time t, choose initial conditions that system is at rest at $x = 0$ when an impulse force is applied: $y(0) = 0 = y'(0)$

Find $y = -e^{-2x} + \left(\frac{1}{2}x^2 - x + 1\right) e^{-x}$

Exercise: Find the extremal displacement from equilibrium and when it occurs.

$y'(x) = 0$, $x \geq 0$ has a unique global maximum point but this equation requires numerical solution.



$f(x) = x e^{-x}$ $\quad x_{max} = 1$, $y_{max} = e^{-1} \approx 0.368$

$y(x)$: $x_{max} \approx 2.56$, $y_{max} \approx 0.127$

E&P3 5.5.9 method of undetermined coefficients: root overlap case

First we determine the trial function for y_p

$$y'' + 2y' - 3y = 1 + xe^{x}$$

$e^{x} \to r=1$, $xe^{x} \to m=2$ (minimum) $\} (D-1)^2(xe^x)=0 \to (D-1)^2[\underbrace{(c_3+c_4x)e^x}_{\text{initial trial function}}]=0$

$1 \to r=0$: $D(1)=0 \to D(c_5)=0$

$y=e^{rx}$: $r^2+2r-3=0$

$(r+3)(r-1)=0$

$r=-3,1$

$e^{rx}=e^{-3x}, e^x$

$y_h = c_1e^{-3x}+c_2e^{x}$

$(D-1)^2D[1+xe^x]=0$

$(D-1)^2D(D+3)(D-1)y=0$

$(D+3)(D-1)^3Dy=0$

$r=-3$, $r=1$, $m=3$, $r=0$

change of multiplicity due to repeated roots on left & right side.

initial trial function BUT c_3e^x satisfies hom DE. multiply by x to get $(c_3x+c_4x^2)e^x$ no term satisfies hom DE stop here

$y_p = c_3(1+c_4x)e^x + c_5$ is trial function

$$y = c_1e^{-3x} + (c_2+c_3x+c_4x^2)e^{x} + c_5$$

$$= y_h + \underbrace{x\,\underbrace{(c_3+c_4x)e^x}_{\text{initial trial function}} + c_5}_{y_p}$$

initial trial function multiplied by x

determining undetermined constants:

$$-3\left[\ y_p = c_5 + (c_3x+c_4x^2)e^x\right]$$

$$+2\left[\ y_p' = 0 + (c_3+2c_4x)e^x + (c_3x+c_4x^2)e^x = [c_3 + (c_3+2c_4)x + c_4x^2]e^x\right]$$

$$+1\left[\ y_p'' = (0+(c_3+2c_4)+2c_4x)e^x + (c_3+(c_3+2c_4)x+c_4x^2)e^x = [(2c_3+2c_4)+(c_3+4c_4)x+c_4x^2]e^x\right]$$

$$y_p''+2y_p'-3y_p = -3c_5 + [2c_3+(c_3+2c_4)+c_3]e^x + [-3c_3+2c_3+4c_4+2c_4+(c_3+2c_4)]e^x + [-3c_4+2c_4+c_4]x^2e^x$$

tedious algebra for sure!

$$= \underbrace{-3c_5}_{=1} + \underbrace{(4c_3+2c_4)}_{=0}e^x + \underbrace{8c_4}_{=1}xe^x + 0x^2e^x = 1+xe^x$$

necessary! otherwise two many equations for too few constants

$c_5=-\frac{1}{3}$, $c_4=\frac{1}{8}$, $c_3=-\frac{1}{2}c_4=-\frac{1}{16}$

$$\boxed{y_p = -\frac{1}{3}+\frac{1}{16}(2x^2-x)} \to \boxed{y = c_1e^{-3x}+c_2e^{x}+-\frac{1}{3}+\frac{1}{16}(2x^2-x)}$$