$\boxed{\text{E2P3 5.5.9}}$ Find $y_p$ :

$$y'' + 2y' - 3y = 1 + xe^x = f(t)$$

$y = e^{rx}$

$r^2 + 2r - 3 = (r-1)(r+3) = 0$

hom DE : $(D-1)(D+3)\, y_h = 0$

$y_h = c_1 e^{-3x} + c_2 e^x$

$r = 1, -3$
$m = 1, 1$

root overlap with RHS

$r = 0$

$e^{rx} = 1$

$(D-0)\, 1 = 0$

gen soln:

$y_p = c_5$
(this term alone)

$\left.\begin{matrix} r = 1 \\ m = 2 \end{matrix}\right\} \underbrace{(D-1)^2 (xe^x) = 0}$

gen soln:

$y_p = (c_3 + c_4 x)\, e^x$
(this term alone)

" If any term in gen soln satisfies hom DE, multiply by x, repeat if any term still satisfies hom DE..."

$c_3$ term, mult by x

$y_p \to x(c_3 + c_4 x)\, e^x$

adding two trial functions

$y_p = (c_3 x + c_4 x^2) e^x + c_5$

$D(D-1)^2 f(t) = 0$
apply to both sides of DE!

$$D(D-1)^2 (D-1)(D+3)\, y = 0$$

$r = 0$
$m = 1$

$r = 1$
$m = 3$

$r = -3$
$m = 1$

gen soln: $y = \underbrace{c_1 e^{-3x}}_{y_h} + \underbrace{(c_2 + c_3 x + c_4 x^2) e^x + c_5}_{y_p}$

Same result (I made sure coefficient indices agreed!)

determine undetermined parameters:

$-3\, [\; y_p = (c_3 x + c_4 x^2) e^x + c_5 \;]$

$2\, [\; y_p' = (c_3 + 2c_4 x) e^x + (c_3 x + c_4 x^2) e^x + 0 = (c_3 + (c_3 + 2c_4)x + c_4 x^2) e^x \;]$

$1\, [\; y_p'' = (c_3 + 2c_4 + 2c_4 x) e^x + (c_3 + (c_3 + 2c_4)x + c_4 x^2) e^x \;]$

$= [(2c_3 + 2c_4) + (c_3 + 4c_4)x + c_4 x^2]\, e^x$

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$y_p'' + 2y_p' - 3y_p = \quad (-3c_3 x - 3c_4 x^2)\, e^x \quad - 3c_5$
$+ (2c_3 \quad + 2(c_3 + 2c_4)x + 2c_4 x^2)\, e^x \quad + 0$
$+ [2c_3 + 2c_4 \quad + (c_3 + 4c_4)x + c_4 x^2]\, e^x \quad + 0$

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$= [\underbrace{(4c_3 + 2c_4)}_{=0} + \underbrace{(0c_3 + 8c_4)}_{1} x + 0x^2] e^x \quad \underbrace{-3c_5}_{=1} = 1 + xe^x$

$c_3 = -\frac{1}{2} c_4 = -\frac{1}{16}$

$c_4 = \frac{1}{8}$

$c_5 = -\frac{1}{3}$

$$\boxed{y_p = \left(-\frac{1}{16} + \frac{x}{8}\right) e^x - \frac{1}{3}}$$