E&P2 5.4.23 : $mx'' + cx' + kx = 0$

This problem deals with a highly simplified model of a car of weight 3200 lb (mass ~~=~~ 100 slugs in fps units) ← $m = 100$

units! ft, lb, sec

Assume that the suspension system acts like a single spring and its shock absorbers satisfy the above equation with appropriate values of the coefficients.

a) Find the stiffness coefficient k of the spring if the car undergoes free vibrations at 80 cycles/min when its shock absorbers are disconnected. ← $\omega_0 = 80 \frac{\text{cycles}}{\text{min}}$ ← when $c = 0$

b) With the shock absorbers connected the car is set into vibration by driving it over a bump, and the resulting damped vibrations have a frequency of 78 cycles/min. ← $\omega = 78 \frac{\text{cycles}}{\text{min}}$

After how long will the time-varying amplitude be 1% of its initial value? ← $A e^{-t/\tau} = .01A$

$e^{-4.6} = .01$

soln

a) $100x'' + \overset{\text{undamped}}{0}x' + kx = 0 \rightarrow x'' + \underbrace{\frac{k}{100}}_{\omega_0^2} x = 0$

$\omega_0^2 \rightarrow k = 100\,\omega_0^2 = 100\left(80 \cdot \frac{2\pi}{60}\right)^2 \approx 7018.4$ lbs/ft ?

$\omega_0 = 80 \frac{\text{cycles}}{\text{min}} = 80 \frac{(2\pi \text{ rad})}{(60 \text{ sec})} = 80\left(\frac{2\pi}{60}\right)\frac{\text{rad}}{\text{sec}} = \frac{8\pi}{3}\frac{\text{rad}}{\text{sec}} \approx 8.168 \frac{\text{rad}}{\text{se}}$

b) $100x'' + cx' + kx = 0$ ← now known

$100r^2 + cr + k = 0$

$r = \frac{-c \pm \sqrt{c^2 - 400k}}{200} = -\frac{c}{200} \pm i\underbrace{\frac{\sqrt{400k - c^2}}{200}}_{\omega} = -k \pm i\omega$

complex roots ("damped vibrations")

given that $\omega = 78 \frac{\text{cycles}}{\text{min}} = \left(\frac{78 \cdot 2\pi}{60}\right)\frac{\text{rad}}{\text{sec}}$

solve for c

$\frac{\sqrt{400k - c^2}}{200} = \left(\frac{78 \cdot 2\pi}{60}\right) \rightarrow 400k - c^2 = 200^2\left(\frac{78 \cdot 2\pi}{60}\right)^2$

$c^2 = 400k - 200^2\left(\frac{78 \cdot 2\pi}{60}\right)^2$

$= 200^2\left(\frac{80 \cdot 2\pi}{60}\right)^2 - 200^2\left(\frac{78 \cdot 2\pi}{60}\right)^2$

$= 200^2\,(80^2 - 78^2)\left(\frac{2\pi}{60}\right)^2$

$c = 200 \cdot \frac{2\pi}{60}\sqrt{80^2 - 78^2} \approx 372.31$

but $\tau = \frac{1}{k} = \frac{200}{c} \approx 0.5372$ sec

$4.6\tau = 2.4739$ sec $\sim$ 2.47 sec

(It takes 4.6 characteristic times to reduce the initial amplitude by a factor of 100)

[note: we did not need to refer to any formulas other than the quadratic formula!]