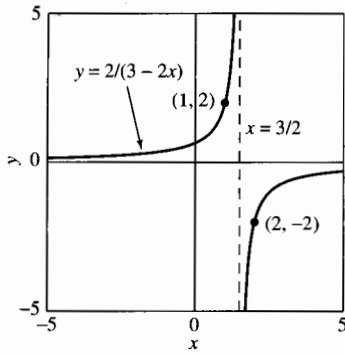


**Solution** We need only find a value of  $C$  so that the solution  $y(x) = 1/(C - x)$  satisfies the initial condition  $y(1) = 2$ . Substitution of the values  $x = 1$  and  $y = 2$  in the given solution yields

$$2 = y(1) = \frac{1}{C - 1},$$

so  $2C - 2 = 1$ , and hence  $C = \frac{3}{2}$ . With this value of  $C$  we obtain the desired solution

$$y(x) = \frac{1}{\frac{3}{2} - x} = \frac{2}{3 - 2x}.$$



**FIGURE 1.1.7.** The solutions of  $y' = y^2$  defined by  $y(x) = 2/(3 - 2x)$ .

Figure 1.1.7 shows the two branches of the graph  $y = 2/(3 - 2x)$ . The left-hand branch is the graph on  $(-\infty, \frac{3}{2})$  of the solution of the given initial value problem  $y' = y^2$ ,  $y(1) = 2$ . The right-hand branch passes through the point  $(2, -2)$  and is therefore the graph on  $(\frac{3}{2}, \infty)$  of the solution of the different initial value problem  $y' = y^2$ ,  $y(2) = -2$ . ■

The central question of greatest immediate interest to us is this: If we are given a differential equation known to have a solution satisfying a given initial condition, how do we actually *find* or *compute* that solution? And, once found, what can we do with it? We will see that a relatively few simple techniques—separation of variables (Section 1.4), solution of linear equations (Section 1.5), elementary substitution methods (Section 1.6)—are enough to enable us to solve a variety of first-order equations having impressive applications.

## 1.1 Problems

In Problems 1 through 12, verify by substitution that each given function is a solution of the given differential equation. Throughout these problems, primes denote derivatives with respect to  $x$ .

- $y' = 3x^2$ ;  $y = x^3 + 7$
- $y' + 2y = 0$ ;  $y = 3e^{-2x}$
- $y'' + 4y = 0$ ;  $y_1 = \cos 2x$ ,  $y_2 = \sin 2x$
- $y'' = 9y$ ;  $y_1 = e^{3x}$ ,  $y_2 = e^{-3x}$
- $y' = y + 2e^{-x}$ ;  $y = e^x - e^{-x}$
- $y'' + 4y' + 4y = 0$ ;  $y_1 = e^{-2x}$ ,  $y_2 = xe^{-2x}$
- $y'' - 2y' + 2y = 0$ ;  $y_1 = e^x \cos x$ ,  $y_2 = e^x \sin x$
- $y'' + y = 3 \cos 2x$ ,  $y_1 = \cos x - \cos 2x$ ,  $y_2 = \sin x - \cos 2x$
- $y' + 2xy^2 = 0$ ;  $y = \frac{1}{1 + x^2}$
- $x^2y'' + xy' - y = \ln x$ ;  $y_1 = x - \ln x$ ,  $y_2 = \frac{1}{x} - \ln x$
- $x^2y'' + 5xy' + 4y = 0$ ;  $y_1 = \frac{1}{x^2}$ ,  $y_2 = \frac{\ln x}{x^2}$
- $x^2y'' - xy' + 2y = 0$ ;  $y_1 = x \cos(\ln x)$ ,  $y_2 = x \sin(\ln x)$

In Problems 13 through 16, substitute  $y = e^{rx}$  into the given differential equation to determine all values of the constant  $r$  for which  $y = e^{rx}$  is a solution of the equation.

- $3y' = 2y$
- $4y'' = y$
- $y'' + y' - 2y = 0$
- $3y'' + 3y' - 4y = 0$

In Problems 17 through 26, first verify that  $y(x)$  satisfies the given differential equation. Then determine a value of the constant  $C$  so that  $y(x)$  satisfies the given initial condition. Use a computer or graphing calculator (if desired) to sketch several typical solutions of the given differential equation, and highlight the one that satisfies the given initial condition.

- $y' + y = 0$ ;  $y(x) = Ce^{-x}$ ,  $y(0) = 2$
- $y' = 2y$ ;  $y(x) = Ce^{2x}$ ,  $y(0) = 3$
- $y' = y + 1$ ;  $y(x) = Ce^x - 1$ ,  $y(0) = 5$
- $y' = x - y$ ;  $y(x) = Ce^{-x} + x - 1$ ,  $y(0) = 10$
- $y' + 3x^2y = 0$ ;  $y(x) = Ce^{-x^3}$ ,  $y(0) = 7$
- $e^y y' = 1$ ;  $y(x) = \ln(x + C)$ ,  $y(0) = 0$
- $x \frac{dy}{dx} + 3y = 2x^5$ ;  $y(x) = \frac{1}{4}x^5 + Cx^{-3}$ ,  $y(2) = 1$
- $xy' - 3y = x^3$ ;  $y(x) = x^3(C + \ln x)$ ,  $y(1) = 17$
- $y' = 3x^2(y^2 + 1)$ ;  $y(x) = \tan(x^3 + C)$ ,  $y(0) = 1$
- $y' + y \tan x = \cos x$ ;  $y(x) = (x + C) \cos x$ ,  $y(\pi) = 0$

In Problems 27 through 31, a function  $y = g(x)$  is described by some geometric property of its graph. Write a differential equation of the form  $dy/dx = f(x, y)$  having the function  $g$  as its solution (or as one of its solutions).

- The slope of the graph of  $g$  at the point  $(x, y)$  is the sum of  $x$  and  $y$ .
- The line tangent to the graph of  $g$  at the point  $(x, y)$  intersects the  $x$ -axis at the point  $(x/2, 0)$ .

29. Every straight line normal to the graph of  $g$  passes through the point  $(0, 1)$ .
30. The graph of  $g$  is normal to every curve of the form  $y = x^2 + k$  ( $k$  is a constant) where they meet.
31. The line tangent to the graph of  $g$  at  $(x, y)$  passes through the point  $(-y, x)$ .

In Problems 32 through 36, write—in the manner of Eqs. (3) through (6) of this section—a differential equation that is a mathematical model of the situation described.

32. The time rate of change of a population  $P$  is proportional to the square root of  $P$ .
33. The time rate of change of the velocity  $v$  of a coasting motorboat is proportional to the square of  $v$ .
34. The acceleration  $dv/dt$  of a Lamborghini is proportional to the difference between 250 km/h and the velocity of the car.
35. In a city having a fixed population of  $P$  persons, the time rate of change of the number  $N$  of those persons who have heard a certain rumor is proportional to the number of those who have not yet heard the rumor.
36. In a city with a fixed population of  $P$  persons, the time rate of change of the number  $N$  of those persons infected with a certain contagious disease is proportional to the product of the number who have the disease and the number who do not.

In Problems 37 through 42, determine by inspection at least one solution of the given differential equation. That is, use your knowledge of derivatives to make an intelligent guess. Then test your hypothesis.

37.  $y'' = 0$                       38.  $y' = y$
39.  $xy' + y = 3x^2$             40.  $(y')^2 + y^2 = 1$
41.  $y' + y = e^x$                 42.  $y'' + y = 0$
43. (a) If  $k$  is a constant, show that a general (1-parameter) solution of the differential equation

$$\frac{dx}{dt} = kx^2$$

is given by  $x(t) = 1/(C - kt)$  where  $C$  is an arbitrary constant.

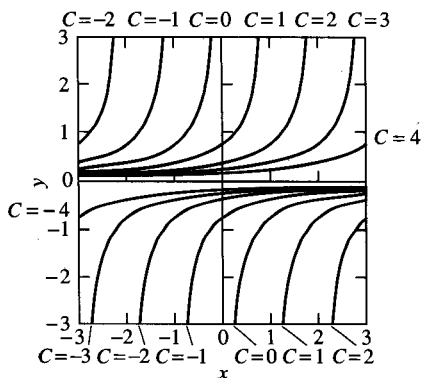


FIGURE 1.1.8. Graphs of solutions of the equation  $dy/dx = y^2$ .

- (b) Determine by inspection a solution of the initial value problem  $x' = kx^2$ ,  $x(0) = 0$ .
44. (a) Continuing Problem 43, assume that  $k$  is positive, and then sketch graphs of solutions of  $x' = kx^2$  with several typical positive values of  $x(0)$ .
- (b) How would these solutions differ if the constant  $k$  were negative?
45. Suppose a population  $P$  of rodents satisfies the differential equation  $dP/dt = kP^2$ . Initially, there are  $P(0) = 2$  rodents, and their number is increasing at the rate of  $dP/dt = 1$  rodent per month when there are  $P = 10$  rodents. How long will it take for this population to grow to a hundred rodents? To a thousand? What's happening here?
46. Suppose the velocity  $v$  of a motorboat coasting in water satisfies the differential equation  $dv/dt = kv^2$ . The initial speed of the motorboat is  $v(0) = 10$  meters per second (m/s), and  $v$  is decreasing at the rate of  $1 \text{ m/s}^2$  when  $v = 5$  m/s. How long does it take for the velocity of the boat to decrease to  $1 \text{ m/s}$ ? To  $\frac{1}{10} \text{ m/s}$ ? When does the boat come to a stop?
47. In Example 7 we saw that  $y(x) = 1/(C - x)$  defines a one-parameter family of solutions of the differential equation  $dy/dx = y^2$ . (a) Determine a value of  $C$  so that  $y(10) = 10$ . (b) Is there a value of  $C$  such that  $y(0) = 0$ ? Can you nevertheless find by inspection a solution of  $dy/dx = y^2$  such that  $y(0) = 0$ ? (c) Figure 1.1.8 shows typical graphs of solutions of the form  $y(x) = 1/(C - x)$ . Does it appear that these solution curves fill the entire  $xy$ -plane? Can you conclude that, given any point  $(a, b)$  in the plane, the differential equation  $dy/dx = y^2$  has exactly one solution  $y(x)$  satisfying the condition  $y(a) = b$ ?
48. (a) Show that  $y(x) = Cx^4$  defines a one-parameter family of differentiable solutions of the differential equation  $xy' = 4y$  (Fig. 1.1.9). (b) Show that

$$y(x) = \begin{cases} -x^4 & \text{if } x < 0, \\ x^4 & \text{if } x \geq 0 \end{cases}$$

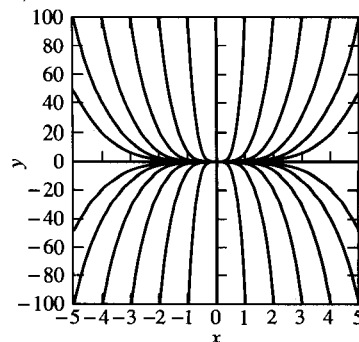


FIGURE 1.1.9. The graph  $y = Cx^4$  for various values of  $C$ .