DEs: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'' = \underbrace{\begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{pmatrix} 0 \\ \cos t \end{pmatrix}$ $\qquad \underline{x}'' = A\underline{x} + \underline{f} \longrightarrow \underline{y}'' = A_B \underline{y} + B^{-1}\underline{f}$

$A: \begin{cases} \lambda = 4, -4 \\ B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad B^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad A_B = B^{-1}AB = \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix} \quad B^{-1}\underline{f} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{pmatrix} 0 \\ \cos t \end{pmatrix} = \begin{bmatrix} 1/2 \cos t \\ 1/2 \cos t \end{bmatrix} \end{cases}$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}'' = \begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 1/2 \cos t \\ 1/2 \cos t \end{bmatrix} = \begin{bmatrix} 4y_1 + \frac{1}{2}\cos t \\ -4y_2 + \frac{1}{2}\cos t \end{bmatrix}$$

$y_1'' - 4y_1 = \frac{1}{2}\cos t \qquad y_{1h}: r^2 - 4 = 0 \quad r = \pm 2, \; e^{rt} = e^{\pm 2t} : \; y_{1h} = c_1 e^{2t} + c_2 e^{-2t}$

$y_2'' + 4y_2 = \frac{1}{2}\cos t \qquad y_{2h}: r^2 + 4 = 0 \quad r = \pm 2i, \; e^{rt} = e^{\pm 2it} \Rightarrow \cos 2t \pm i \sin 2t : \; y_{2h} = c_3 \cos 2t + c_4 \sin 2t$

$y_{1p} = c_5 \cos t \qquad y_{1p}'' - 4y_{1p} = -c_5 \cos t - 4c_5 \cos t = -5c_5 \cos t = \frac{1}{2}\cos t \qquad c_5 = -\frac{1}{10}$

$y_{2p} = c_6 \cos t \qquad y_{2p}'' + 4y_{2p} = -c_6 \cos t + 4c_6 \cos t = +3c_6 \cos t = \frac{1}{2}\cos t \quad \cdot \; c_6 = +\frac{1}{6}$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} c_1 e^{2t} + c_2 e^{-2t} & -\frac{1}{10}\cos t \\ c_3 \cos 2t + c_4 \sin 2t & +\frac{1}{6}\cos t \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = B\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = (c_1 e^{2t} + c_2 e^{-2t})\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (c_3 \cos 2t + c_4 \sin 2t)\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{10}\cos t \\ +\frac{1}{6}\cos t \end{bmatrix}}$$

$$\begin{bmatrix} \left(-\frac{1}{10} - \frac{1}{6}\right)\cos 2t \\ \left(-\frac{1}{10} + \frac{1}{6}\right)\cos 2t \end{bmatrix} = \frac{\cos t}{15}\begin{bmatrix} -4 \\ +1 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 e^{2t} + c_2 e^{-2t} - (c_3 \cos 2t + c_4 \sin 2t) - \frac{4}{15}\cos t \\ c_1 e^{2t} + c_2 e^{-2t} + (c_3 \cos 2t + c_4 \sin 2t) + \frac{1}{15}\cos t \end{bmatrix}$$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2c_1 e^{2t} - 2c_2 e^{-2t} & -(-2c_3 \sin 2t + 2c_4 \cos 2t) + \frac{4}{15}\sin t \\ 2c_1 e^{2t} - 2c_2 e^{-2t} & +(-2c_3 \sin 2t + 2c_4 \cos 2t) - \frac{1}{15}\sin t \end{bmatrix}$$

ICs: $\begin{cases} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} c_1 + c_2 - c_3 - 4/15 \\ c_1 + c_2 + c_3 + 1/15 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 2c_1 - 2c_2 & -2c_4 \\ 2c_1 - 2c_2 & +2c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{cases}$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & -2 & 0 & -2 \\ 2 & -2 & 0 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 + 4/15 \\ -1/15 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 3/10 \\ 3/10 \\ -2/3 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{10}(e^{2t} + e^{-2t}) & + \frac{2}{3}\cos 2t & - \frac{4}{15}\cos t \\ \frac{3}{10}(e^{2t} + e^{-2t}) & - \frac{2}{3}\cos 2t & + \frac{1}{15}\cos t \end{bmatrix}$$

$c_1 = c_2 = \frac{3}{10}$

or $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = B\begin{bmatrix} (c_1 + c_2) - \frac{1}{10} \\ (c_3 + c_4) + \frac{1}{6} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{pmatrix} c_1 + c_2 - \frac{1}{10} \\ c_3 + c_4 + \frac{1}{6} \end{pmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$

$c_1 + c_2 = \frac{1}{2} + \frac{1}{10} = 3/5$

$c_3 + c_4 = -\frac{1}{2} - \frac{1}{6} \to c_3 = -\frac{2}{3}$

$\begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = B\begin{bmatrix} 2c_1 - 2c_2 \\ +2c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{pmatrix} 2(c_1 - c_2) \\ 2c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad c_1 - c_2 = 0, \quad c_4 = 0$

not necessary; just a clever observation

$\boxed{\text{a bit sloppy, notice my sign error that messed up my first soln attempt}}$