1st order linear homogeneous DE system: purely imaginary eigenvectors

$x_1' = x_1 - 5x_2$
$x_2' = x_1 - x_2$

$x_1(0) = 1,\ x_2(0) = 2$

$$\to \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -5 \\ 1 & -1-\lambda \end{vmatrix} = (\lambda-1)(\lambda+1) + 5 = \lambda^2 + 4 = 0$$

$$\lambda = \pm 2i$$

$$\lambda = 2i: \quad \begin{pmatrix} 1-2i & -5 \\ 1 & -1-2i \end{pmatrix} \xrightarrow{\text{rref}} \begin{array}{cc} L & F \\ \begin{bmatrix} 1 & -(1+2i) \\ 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$x_1 = (1+2i)t$
$x_2 = t$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (1+2i)t \\ t \end{pmatrix} = t \underbrace{\begin{bmatrix} 1+2i \\ 1 \end{bmatrix}}_{E_1} \to \quad \lambda = -2i: \quad \bar{E}_1 = \begin{bmatrix} 1-2i \\ 1 \end{bmatrix}$$

$$\underline{B} = \begin{bmatrix} 1+2i & 1-2i \\ 1 & 1 \end{bmatrix} \qquad \underline{x} = \underline{B}\,\underline{y}, \quad \underline{y} = \underline{B}^{-1}\underline{x}, \quad A_B = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}$$

$$\underline{x}' = A\underline{x} \longrightarrow \underline{y}' = A_B\underline{y} \qquad \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \to \begin{array}{l} y_1' = 2iy_1 \\ y_2' = -2iy_2 \end{array} \quad \begin{array}{l} y_1 = c_1 e^{2it} \\ y_2 = c_2 e^{-2it} \end{array}$$

$$\underline{x} = \begin{bmatrix} 1+2i & 1-2i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{2it} \\ c_2 e^{-2it} \end{bmatrix} = c_1\, e^{2it} \underbrace{\begin{bmatrix} 1+2i \\ 1 \end{bmatrix}} + c_2\, \underbrace{e^{-2it} \begin{bmatrix} 1-2i \\ 1 \end{bmatrix}}$$

complex conjugate basis solutions

complex coefficients

find real & imag parts → real basis solutions

$$e^{2it}\begin{bmatrix} 1+2i \\ 1 \end{bmatrix} = \begin{bmatrix} (1+2i)(\cos 2t + i \sin 2t) \\ 1 \quad (\cos 2t + i \sin 2t) \end{bmatrix} = \begin{bmatrix} \cos 2t + i \sin 2t + 2i\cos 2t - 2\sin 2t \\ \cos 2t + i \sin 2t \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2t - 2\sin 2t + i(\sin 2t + 2\cos 2t) \\ \cos 2t + i \sin 2t \end{bmatrix} = \underbrace{\begin{bmatrix} \cos 2t - 2\sin 2t \\ \cos 2t \end{bmatrix}}_{\underline{X}_1(t)} + i \underbrace{\begin{bmatrix} \sin 2t + 2\cos 2t \\ \sin 2t \end{bmatrix}}_{\underline{X}_2(t)}$$

$\underline{X}_1(t) \leftrightarrow \underline{X}_2(t)$ real basis

$$\underline{x} = c_1 \underline{X}_1(t) + c_2 \underline{X}_2(t) = c_1 \begin{bmatrix} \cos 2t - 2\sin 2t \\ \cos 2t \end{bmatrix} + c_2 \begin{bmatrix} \sin 2t + 2\cos 2t \\ \sin 2t \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (c_1 + 2c_2)\cos 2t + (-2c_1 + c_2)\sin 2t \\ c_1 \cos 2t + c_2 \sin 2t \end{bmatrix} \begin{pmatrix} \text{gen} \\ \text{soln} \end{pmatrix} = \begin{bmatrix} \cos 2t - 9/2 \sin 2t \\ 2\cos 2t - 1/2 \sin 2t \end{bmatrix} \begin{pmatrix} \text{IVP} \\ \text{soln} \end{pmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2 \\ c_1 \end{bmatrix} \xrightarrow{\text{solve}} \begin{array}{l} c_1 = 2 \\ c_2 = \frac{1-c_1}{2} = -\frac{1}{2} \end{array}$$

$$c_1 + 2c_2 = 1, \quad -2c_1 + c_2 = -2(2) + (-\tfrac{1}{2}) = -9/2$$

phase shifted cosine form:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{85}}{2} \cos(2t + \arctan(9/2)) \\ \frac{\sqrt{17}}{2} \cos(2t + \arctan(1/4)) \end{bmatrix} = \frac{\sqrt{17}}{2} \begin{bmatrix} \sqrt{5} \cos(2t + \overbrace{\arctan(1/4)}^{0.2450} + \overbrace{\arctan(2)}^{1.1071}) \\ 1 \cos(2t + \arctan(1/4)) \end{bmatrix}$$

1.3521

$$\begin{bmatrix} 1+2i \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{5}\, e^{(\arctan 2) i} \\ 1 \end{bmatrix}$$

polar form eigenvector

← gives relative amplitude factor and relative phase shift of two oscillations.

eigenvalue gives frequency of oscillation

Initial conditions set overall amplitude & phase shift