1st order linear homogeneous DE system : real eigenvalues

$$\underline{A} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \qquad \lambda = 5, -1 \qquad \underline{B} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \qquad B^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \qquad A_B = B^{-1}AB = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\underbrace{\vec{b}_1}\ \underbrace{\vec{b}_2}$$

$$\underline{x}' = \underline{A}\underline{x} \qquad \frac{dx_1}{dt} = x_1 + 4x_2 \qquad \frac{dx_2}{dt} = 2x_1 + 3x_2 \qquad \xrightarrow[\underline{y} = \underline{B}^{-1}\underline{x}]{\underline{x} = \underline{B}\underline{y}} \qquad \underline{y}' = \underline{A}_B\underline{y}$$

$$\frac{dy_1}{dt} = 5y_1 \rightarrow y_1 = c_1 e^{5t}$$

$$\frac{dy_2}{dt} = -y_2 \rightarrow y_2 = c_2 e^{-t}$$

$$\downarrow$$

$$\underline{x}(0) = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \text{ — } x_1(0) = -2,\ x_2(0) = 4 \qquad \underline{y}(0) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \longleftarrow y_1(0) = c_1,\ y_2(0) = c_2$$

$$\underline{x}(0) = \underline{B}\,\underline{y}(0) \rightarrow \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \underline{B}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \underline{B}^{-1}\begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\boxed{\text{general soln:}} \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 e^{5t} \\ c_2 e^{-t} \end{bmatrix} = \begin{bmatrix} c_1 e^{5t} - 2c_2 e^{-t} \\ c_1 e^{5t} + c_2 e^{-t} \end{bmatrix}$$

$$= c_1 \underset{\substack{\uparrow \\ \text{growth along } \vec{b}_1}}{e^{5t}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \underset{\substack{\uparrow \\ \text{decay along } \vec{b}_2}}{e^{-t}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

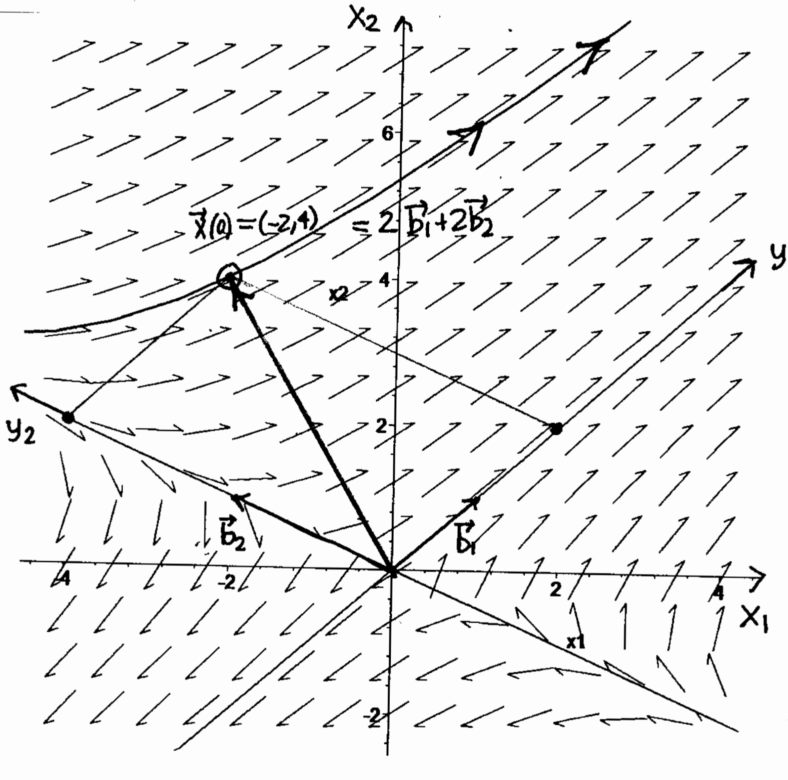
eigenvector entries set relative initial values for variables in each "mode" of exponential behavior

$$\boxed{\text{IVP soln:}} \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2e^{5t} - 4e^{-t} \\ 2e^{5t} + 2e^{-t} \end{bmatrix}$$

$$= 2e^{5t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2e^{-t}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Initial data sets the "mixture" coefficients of the combination of modes

the eigenvalues are the exponential rate coefficients for each mode



the decaying mode $(\lambda = -1)$ decays to 1% of its original value in the time interval $t = 0..4.6$ during which the growing mode $(\lambda = 5)$ grows by a factor $e^{5(4.6)} \approx 10^{10}$ (very big)

1st order linear homogeneous DE system: complex eigenvalues

$\lambda = -1+2i, -1-2i$ (real part is exp. rate coefficient, imag part is frequency)

$$\underline{A} = \begin{bmatrix} 1 & 2 \\ -4 & -3 \end{bmatrix} \qquad \underline{B} = \begin{bmatrix} \underbrace{\begin{matrix} -\frac{1}{2}-\frac{1}{2}i \\ 1 \end{matrix}}_{\underline{b_1}} & \underbrace{\begin{matrix} -\frac{1}{2}+\frac{1}{2}i \\ 1 \end{matrix}}_{\underline{b_2}=\overline{\underline{b_1}}} \end{bmatrix} \qquad \underline{B}^{-1} = \begin{bmatrix} i & \frac{1}{2}+\frac{1}{2}i \\ -i & \frac{1}{2}-\frac{1}{2}i \end{bmatrix} \qquad A_B = B^{-1}AB = \begin{bmatrix} -1+2i & 0 \\ 0 & -1-2i \end{bmatrix}$$

$$\underline{x}' = \underline{A}\,\underline{x} \qquad \begin{matrix} \frac{dx_1}{dt} = x_1 + 2x_2 \\ \frac{dx_2}{dt} = -4x_1 - 3x_2 \end{matrix} \xrightarrow[\underline{y}=\underline{B}^{-1}\underline{x}]{\underline{x}=\underline{B}\,\underline{y}} \underline{y}' = A_B\,\underline{y} \qquad \begin{matrix} \frac{dy_1}{dt} = (-1+2i)y_1 \rightarrow y_1 = c_1 e^{(-1+2i)t} \\ \frac{dy_2}{dt} = (-1-2i)\,y_2 \rightarrow y_2 = c_2 e^{(-1-2i)t} \end{matrix}$$

$$\downarrow$$

$$y_1(0) = c_1\,,\ y_2(0) = c_2$$

$$\underline{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad x_1(0) = 1\,,\ x_2(0) = 2 \qquad \underline{y}(0) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \longleftarrow$$

$$\underline{x}(0) = \underline{B}\,\underline{y}(0) \rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix} = B \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = B^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+2i \\ 1-2i \end{bmatrix}$$

$\rightarrow c_2 = \overline{c_1}$ for $\underline{x}$ to be real

$$\boxed{\text{general soln:}} \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}(1+i) & -\frac{1}{2}(1-i) \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{-t} e^{2ti} \\ c_2 e^{-t} e^{-2ti} \end{bmatrix} = \underbrace{c_1 e^{-t} e^{2ti} \begin{bmatrix} -\frac{1}{2}(1+i) \\ 1 \end{bmatrix}}_{\text{complex soln} \longrightarrow} + \underbrace{c_2 e^{-t} e^{-2ti} \begin{bmatrix} -\frac{1}{2}(1-i) \\ 1 \end{bmatrix}}_{\text{its complex conjugate}}$$

Re and Im parts:

$$e^{-t} e^{2ti} \begin{bmatrix} -\frac{1}{2}(1+i) \\ 1 \end{bmatrix} = e^{-t} \begin{bmatrix} -\frac{1}{2}(1+i)(\cos 2t + i \sin 2t) \\ \cos 2t + i \sin 2t \end{bmatrix} = e^{-t} \begin{bmatrix} -\frac{1}{2}\cos 2t + \frac{1}{2}\sin 2t \\ \cos 2t \end{bmatrix}$$

$$+ i e^{-t} \begin{bmatrix} -\frac{1}{2}\sin 2t - \frac{1}{2}\cos 2t \\ \sin 2t \end{bmatrix}$$

so

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a\ e^{-t} \begin{bmatrix} -\frac{1}{2}\cos 2t + \frac{1}{2}\sin 2t \\ \cos 2t \end{bmatrix} + b\ e^{-t} \begin{bmatrix} -\frac{1}{2}\sin 2t - \frac{1}{2}\cos 2t \\ \sin 2t \end{bmatrix}$$

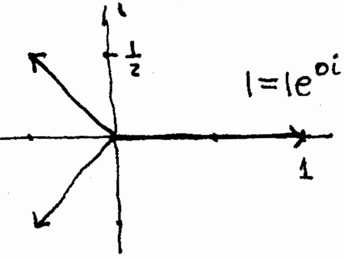
explicitly real general soln

$$\boxed{\text{IVP soln:}} \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (1+2i)\, e^{-t} e^{2it} \begin{bmatrix} -\frac{1}{2}(1+i) \\ 1 \end{bmatrix} + \text{c.c.} = e^{-t} e^{2it} \begin{bmatrix} \frac{1}{2}(1-3i) \\ 1+2i \end{bmatrix} + \text{c.c.} = e^{-t} \begin{bmatrix} \frac{1}{2}(1-3i)(\cos 2t + i \sin 2t) \\ (1+2i)(\cos 2t + i \sin 2t) \end{bmatrix} + \text{c.c.}$$

$$= e^{-t} \begin{bmatrix} \cos 2t + 3 \sin 2t \\ 2\cos 2t - 4 \sin 2t \end{bmatrix} = 2\ e^{-t} \begin{bmatrix} -\frac{1}{2}\cos 2t + \frac{1}{2}\sin 2t \\ \cos 2t \end{bmatrix} - 4\ e^{-t} \begin{bmatrix} -\frac{1}{2}\sin 2t - \frac{1}{2}\cos 2t \\ \sin 2t \end{bmatrix}$$

$$= e^{-t} \begin{bmatrix} \sqrt{10}\ \cos(2t - \delta_1) \\ 2\sqrt{5}\ \cos(2t - \delta_2) \end{bmatrix} \qquad \begin{matrix} \delta_1 = \arctan 3 \\ \delta_2 = -\arctan 2 \end{matrix}$$

$$-\frac{1}{2} + \frac{i}{2} = \frac{1}{\sqrt{2}} e^{i\frac{3\pi}{4}} \qquad 1 = 1e^{0i}$$



$$-\frac{1}{2} - \frac{i}{2} = \frac{1}{\sqrt{2}} e^{-i\frac{3\pi}{4}}$$

$$\text{polar form of eigenvector entries:} \quad \underline{b_1} = \begin{bmatrix} \frac{1}{\sqrt{2}} e^{-i3\pi/4} \\ 1\, e^{i0} \end{bmatrix}$$

relative amplitudes ⟶ ↑ ↑ relative phase shifts

$$\text{polar form of initial data determined constants:} \quad 2\overline{c_1} = 2c_2 = 2 - 4i = a + ib = 2\sqrt{5}\, e^{-i \arctan 2}$$

effect on soln:

$$\underline{x} = \text{Re}\left(2c_1\, e^{-t} e^{2ti}\, \underline{b_1}\right) = \text{Re}\left\{ \underbrace{\left(2\sqrt{5}\, e^{i \arctan 2}\right)}_{\text{sets overall amplitude, phase shift}} e^{-t} e^{2ti} \begin{bmatrix} \frac{1}{\sqrt{2}} e^{-\frac{3\pi}{4} i} \\ 1\, e^{0i} \end{bmatrix} \right\}$$

↑ fixes amplitude ratios

↑ ⟵ fixes relative phase shifts $\delta_1 - \delta_2 = \frac{3\pi}{4}$

$$= \text{Re}\ e^{-t} \begin{bmatrix} \sqrt{10}\ e^{(2t + \arctan 2 - 3\pi/4)i} \\ 2\sqrt{5}\ e^{(2t + \arctan 2 - 0\ )i} \end{bmatrix} = \begin{bmatrix} \sqrt{10}\ \cos(2t + \arctan 2 - 3\pi/4) \\ 2\sqrt{5}\ \cos(2t + \arctan 2\ -\ 0\ ) \end{bmatrix}$$