$\vec{x}'' = A\vec{x}$ IVP (2x2 coefficient matrix)

$$\left.\begin{array}{ll} x_1'' = (-5x_1+3x_2)/2, & x_1(0)=1,\ x_1'(0)=0 \\ x_2'' = (3x_1-5x_2)/2, & x_2(0)=0,\ x_2'(0)=1 \end{array}\right\} \text{IVP}$$

$$\begin{bmatrix}x_1\\x_2\end{bmatrix}'' = \underbrace{\begin{bmatrix}-5 & 3\\ 3 & -5\end{bmatrix}}_{A}\begin{bmatrix}x_1\\x_2\end{bmatrix}, \quad \begin{bmatrix}x_1(0)\\x_2(0)\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}, \quad \begin{bmatrix}x_1'(0)\\x_2'(0)\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$$ matrix form

$$0 = \det(A-\lambda I) = \lambda^2+5\lambda+4 = (\lambda+1)(\lambda+4)$$

$$\lambda = -1, -4 \quad \to \cdots \quad B = \langle \vec{b}_1 | \vec{b}_2 \rangle = \begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$$

$$B^{-1} = \frac{1}{2}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix} \quad A_B = B^{-1}AB = \begin{bmatrix}-1 & 0\\ 0 & -4\end{bmatrix}$$

$$\vec{x} = B\vec{y} = y_1\vec{b}_1 + y_2\vec{b}_2, \quad \vec{y} = B^{-1}\vec{x}$$

diagonalization, new variables

$$\vec{x}'' = A\vec{x} \longrightarrow B^{-1}(B\vec{y})' = BAB\vec{y}$$

$$\vec{y}'' = A_B\vec{y}$$

$$\begin{bmatrix}y_1\\y_2\end{bmatrix}'' = \begin{bmatrix}-1 & 0\\ 0 & -4\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix} = \begin{bmatrix}-y_1\\-4y_2\end{bmatrix}$$

$$y_1'' = -y_1 \qquad y_1 = c_1\cos t + c_2\sin t$$
$$y_2'' = -4y_2 \qquad y_2 = c_3\cos 2t + c_4\sin 2t$$

decoupling DEs, solving them

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = B\begin{bmatrix}c_1\cos t + c_2\sin t\\ c_3\cos 2t + c_4\sin 2t\end{bmatrix}$$

$\downarrow$ solving ICs

$$\begin{bmatrix}x_1'\\x_2'\end{bmatrix} = B\begin{bmatrix}-c_1\sin t + c_2\cos t\\ -2c_3\sin 2t + 2c_4\cos 2t\end{bmatrix}$$

oops, not needed!

$$\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}x_1(0)\\x_2(0)\end{bmatrix} = B\begin{bmatrix}c_1\\c_3\end{bmatrix}, \quad \begin{bmatrix}c_1\\c_3\end{bmatrix} = B^{-1}\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix}$$

$$\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}x_1'(0)\\x_2'(0)\end{bmatrix} = B\begin{bmatrix}c_2\\2c_4\end{bmatrix}, \quad \begin{bmatrix}c_2\\2c_4\end{bmatrix} = B^{-1}\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}$$

$$\begin{bmatrix}c_2\\c_4\end{bmatrix} = \begin{bmatrix}1/2\\1/4\end{bmatrix}$$

$$\begin{bmatrix}y_1\\y_2\end{bmatrix} = \begin{bmatrix}\frac{1}{2}(\cos t + \sin t)\\ \frac{1}{4}(-2\cos 2t + \sin 2t)\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\cos(t - \pi/4)\\ \frac{\sqrt{5}}{4}\cos(2t - (\pi - \arctan 1/2))\end{bmatrix}$$

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}(\cos t + \sin t)/2\\ (-2\cos 2t + \sin 2t)/4\end{bmatrix}$$

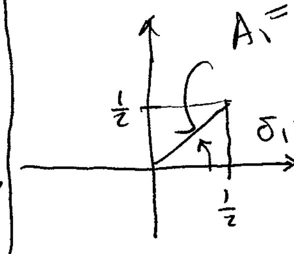
$$= \begin{bmatrix}\frac{1}{2}\cos t + \frac{1}{2}\sin t + \frac{1}{2}\cos 2t - \frac{1}{4}\sin 2t\\ \frac{1}{2}\cos t - \frac{1}{2}\sin t - \frac{1}{2}\cos 2t + \frac{1}{4}\sin 2t\end{bmatrix}$$

$$\vec{x} = \frac{1}{\sqrt{2}}\cos(t-\delta_1)\,\vec{b}_1 + \frac{\sqrt{5}}{4}\cos(2t-\delta_2)\,\vec{b}_2$$

slow mode $\qquad$ fast mode

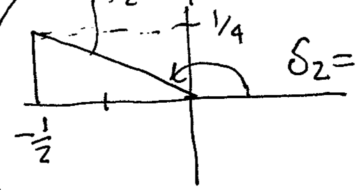
phase-shifted cosine calculation

$$A_1 = \frac{1}{2}\sqrt{2} = \frac{1}{\sqrt{2}} \approx .71$$

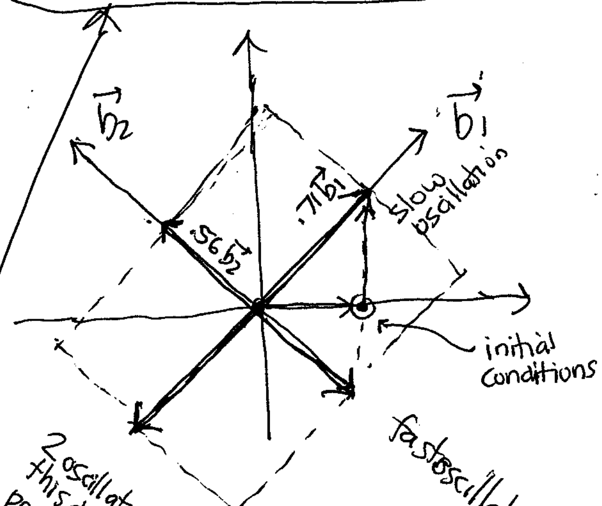


$\delta_1 = \pi/4$

$$A_2 = \frac{1}{4}\sqrt{4+1} = \sqrt{5}/4 \approx .56$$



$\delta_2 = \pi - \arctan\frac{1}{2}$



2 oscillations in this direction per 1 oscillation in the other so we have to get a figure 8 like curve