driven damped harmonic oscillator system

SPECIFIC EXAMPLE: (barely underdamped)

$$\underset{k_0}{y''+4}y'+\underset{\omega_0^2}{5}y = \underset{B_0}{10}\cos \underset{\omega_0}{3}t = f(t) \longleftrightarrow e^{\pm 3t}$$

$r = \pm 3i$

$\tau_0 = 1/4$ $\quad \omega_0 = \sqrt{5} \approx 2.24 < \omega_0$

$Q = \frac{\sqrt{5}}{4} \approx 0.56$ $\quad T_0 = 2\pi/\sqrt{5} \approx 2.81$

$y = e^{rt}$: $r^2+4r+5=0$

$$r = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$$

no root overlap

$e^{rt} = e^{-2t}(\cos t \pm i \sin t)$

$y_h = \underbrace{e^{-2t}}(c_1\cos t + c_2 \sin t)$

dies away so y_h = "transient"

$5\,[y_p = c_3\cos 3t + c_4 \sin 3t]$

$4\,[y_p' = -3c_3\sin 3t + 3c_4\cos 3t]$

$1\,[y_p'' = -9c_3\cos 3t - 9c_4\sin 3t]$

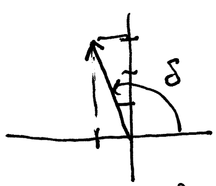
$$y_p''+4y_p'+5y_p = [\underbrace{(5-9)c_3+12c_4}_{=10}]\cos 3t = 10\cos 3t$$
$$+ [\underbrace{-12c_3+(5-9)c_4}_{=0}]\sin 3t$$

$$\begin{bmatrix}(5-9) & 12\\ -12 & 5-9\end{bmatrix}\begin{bmatrix}c_3\\c_4\end{bmatrix} = \begin{bmatrix}10\\0\end{bmatrix} \rightarrow \begin{bmatrix}-4 & 12\\ -12 & -4\end{bmatrix}\begin{bmatrix}c_3\\c_4\end{bmatrix} = \begin{bmatrix}10\\0\end{bmatrix}$$

$$\begin{bmatrix}c_3\\c_4\end{bmatrix} = \frac{\begin{bmatrix}-4 & -12\\ 12 & -4\end{bmatrix}}{16+144}\begin{bmatrix}10\\0\end{bmatrix} = \frac{10}{160}\begin{bmatrix}-4\\12\end{bmatrix} = \begin{bmatrix}-1/4\\3/4\end{bmatrix}$$

$y_p = \frac{1}{4}(-\cos 3t + 3\sin 3t)$

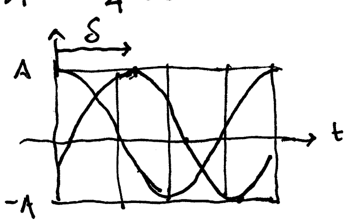
$\frac{1}{4}(-1,3)$:



$A = \frac{1}{4}\sqrt{1+9} = \frac{\sqrt{10}}{4} \approx 0.79$

$\delta = \pi - \arctan 3 \approx 108° \approx 0.30$ cycles

$y_p = \frac{\sqrt{10}}{4}\cos(3t - \pi + \arctan 3)$ "response function"



peaks later in time than "driving function" $f(t)$

$y = y_h + y_p = e^{-2t}(c_1\cos t + c_2 \sin t)$ "transient"

$+ \frac{1}{4}(-\cos 3t + 3\sin 3t)$

"steady state solution"

$\frac{A\omega_0^2}{B_0} = \frac{9\sqrt{10}/4}{10} = 9/4\sqrt{10} \approx 0.71$ amplitude ratio: response / driving function

GENERAL CASE:

$y'' + k_0 y' + \omega_0^2 y = B_0 \cos \omega t \rightarrow r = \pm i\omega$

$y = e^{rt}$: $r^2 + k_0 r + \omega_0^2 = 0$

$r = \frac{-k_0 \pm \cdots}{2} \longleftarrow$ no root overlap

y_h = "transient," real roots negative, complex roots have negative real part so exponentials cause y_h to die away.

$\omega_0^2\,[y_p = c_3\cos\omega t + c_4\sin\omega t]$

$k_0\,[y_p' = -\omega c_3 \sin\omega t + \omega c_4\cos\omega t]$

$1\,[y_p'' = -\omega^2 c_3\cos\omega t - \omega^2 c_4 \sin\omega t]$

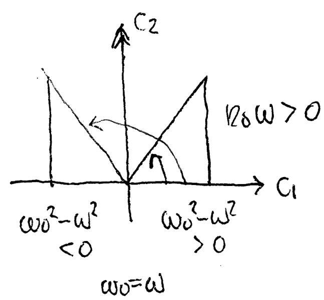
$$y_p'' + k_0 y_p' + \omega_0^2 y = [\underbrace{(\omega_0^2-\omega^2)c_3 + (k_0\omega)c_4}_{=B_0}]\cos\omega t$$
$$+ [\underbrace{-k_0\omega c_3 + (\omega_0^2-\omega^2)c_4}_{=0}]\sin\omega t$$
$$= B_0\cos\omega t$$

$$\begin{bmatrix}(\omega_0^2-\omega^2) & k_0\omega\\ -k_0\omega & (\omega_0^2-\omega^2)\end{bmatrix}\begin{bmatrix}c_3\\c_4\end{bmatrix} = \begin{bmatrix}B_0\\0\end{bmatrix}$$

$$\begin{bmatrix}c_3\\c_4\end{bmatrix} = \frac{1}{(\omega_0^2-\omega^2)^2 + k_0^2\omega^2}\begin{bmatrix}\omega_0^2-\omega^2 & -k_0\omega\\ +k_0\omega & \omega_0^2-\omega^2\end{bmatrix}\begin{bmatrix}B_0\\0\end{bmatrix}$$

$$= \frac{B_0}{(\omega_0^2-\omega^2)^2 + k_0^2\omega^2}\begin{bmatrix}\omega_0^2-\omega^2\\ +k_0\omega\end{bmatrix}$$

$$y_p = \frac{B_0}{(\omega_0^2-\omega^2)^2 + k_0^2\omega^2}[(\omega_0^2-\omega^2)\cos\omega t + k_0\omega\sin\omega t]$$



$$A = \frac{B_0}{[(\ldots)^2 + k_0^2\omega^2]}\sqrt{(\omega_0^2-\omega^2)^2 + k_0^2\omega^2} \quad \longleftarrow \text{same}$$

$$= \frac{B_0}{\sqrt{(\omega_0^2-\omega^2)^2 + k_0^2\omega^2}}$$

$$\frac{A\omega_0^2}{B_0} = \frac{\omega_0^2}{\sqrt{(\omega_0^2-\omega^2)^2 + k_0^2\omega^2}}$$

amplitude ratio response/driving function

$\tan\delta = \frac{k_0\omega}{\omega_0^2-\omega^2}$

$$\begin{cases} > 0 \text{ if } \omega_0 > \omega & \text{acute}\\ = 0 \text{ if } \omega_0 = \omega & \pi/2\\ < 0 \text{ if } \omega_0 < \omega & \text{obtuse}\end{cases}$$

limit as $\omega \to 0$ is 1
zero frequency limit comparison (constant driving function)