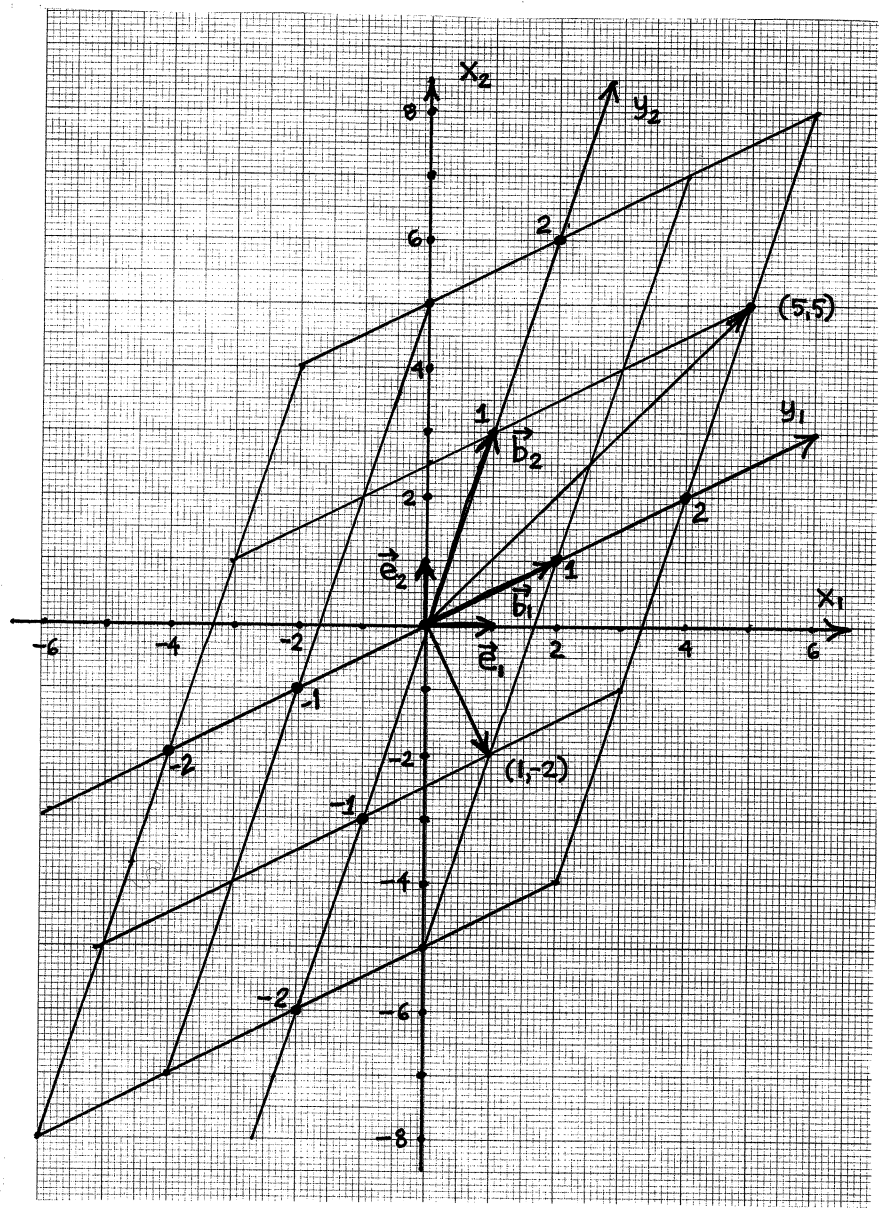
Nonstandard coordinates on $\mathbb{R}^2$



- old basis:
  $\vec{e}_1 = \langle 1,0\rangle$, $\vec{e}_2 = \langle 0,1\rangle$
  new basis:
  $\vec{b}_1 = \langle 2,1\rangle$, $\vec{b}_2 = \langle 1,3\rangle$

basis changing matrix:

$B = \text{augment}(\vec{b}_1, \vec{b}_2) = \langle \vec{b}_1 | \vec{b}_2 \rangle$

$= \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ cols = old coords of new basis vectors

inverse:

$B^{-1} = \frac{1}{5}\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{bmatrix}$

cols = new coords of old basis vectors

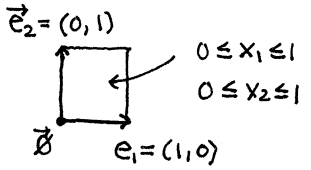
2 example vectors:

$\langle 5,5\rangle = 5\vec{e}_1 + 5\vec{e}_2$
$= 2\vec{b}_1 + 1\vec{b}_2$

$x_1 = 5,\ x_2 = 5$
$y_1 = 2,\ y_2 = 1$

$\langle 1,-2\rangle = 1\vec{e}_1 - 2\vec{e}_2$
$= 1\vec{b}_1 - 1\vec{b}_2$

$x_1 = 1,\ x_2 = -2$
$y_1 = 1,\ y_2 = -1$



The graph paper has the unit coordinate grid for the standard cartesian coordinates $\{x_1, x_2\}$ marked by bold line 1cm tickmarks (refined to 1mm tenth tickmarks).

The unit coordinate rectangle at the origin tiles the plane to make this grid.

The new unit coordinate grid is a tiling of the plane by the unit coordinate "rectangle" $0 \le y_1 \le 1$, $0 \le y_2 \le 1$ (actually a parallelogram) at the origin. The region shown is $-2 \le y_1 \le 2$, $-2 \le y_2 \le 2$.

The relationship between the two coordinate systems

$$\vec{X} = \langle x_1, x_2\rangle = x_1\vec{e}_1 + x_2\vec{e}_2 = y_1\vec{b}_1 + y_2\vec{b}_2 = y_1\langle 2,1\rangle + y_2\langle 1,3\rangle$$

or $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = y_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y_2\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ $\longleftrightarrow$ $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

or $\underset{\text{"old"}}{X} = \underset{\text{"new"}}{BY}$ $\longleftrightarrow$ $\underset{\text{"new"}}{Y} = \underset{\text{"old"}}{B^{-1}X}$

or $x_1 = 2y_1 + y_2$, $x_2 = 1y_1 + 3y_2$ — linear change of coordinates — $y_1 = (3x_1 - x_2)/5$, $y_2 = (-x_1 + 2x_2)/5$

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with(plots) : display(plot(0, x =-10 ..10, y =-10 ..10, axis = [gridlines = [21, color = black]], scaling
    = constrained, thickness = 2, color = black), plot( [ [0, -10], [0, 10] ], thickness = 2, color = black) )
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