change of basis on a plane subspace (example)

$\vec{v}_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}1\\3\\2\end{bmatrix}, \vec{v}_3 = \begin{bmatrix}2\\5\\5\end{bmatrix}$ Find a basis for span$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

$$\langle \vec{v}_1 | \vec{v}_2 | \vec{v}_3 \rangle = \begin{bmatrix}1&1&2\\2&3&5\\3&2&5\end{bmatrix} \xrightarrow{rref} \begin{array}{ccc} L & L & F \end{array} \begin{bmatrix}1&0&1\\0&1&1\\0&0&0\end{bmatrix}$$

third column = sum of first two, throw out third vector of original set to get linearly ind. subset & hence a basis of the plane spanned by these three vectors: $\{\vec{v}_1, \vec{v}_2\}$ basis

Any vector in the plane can be uniquely expressed as

$$\vec{x} = z_1\vec{v}_1 + z_2\vec{v}_2 \quad \text{or} \quad \langle \vec{v}_1|\vec{v}_2\rangle \begin{bmatrix}z_1\\z_2\end{bmatrix} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \leftrightarrow \begin{bmatrix}1&1\\2&3\\3&2\end{bmatrix}\begin{bmatrix}z_1\\z_2\end{bmatrix} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \to \text{solve:}$$

(explicit representation of points)

$$\begin{bmatrix}1&1&x_1\\2&3&x_2\\3&2&x_3\end{bmatrix} \to \begin{bmatrix}1&1&x_1\\0&1&x_2-2x_1\\0&-1&x_3-3x_1\end{bmatrix} \to \begin{bmatrix}1&1&x_1\\0&1&x_2-2x_1\\0&0&x_3+x_2-5x_1\end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$; $R_3 \to R_3 + R_2$

inconsistent unless $-5x_1 + x_2 + x_3 = 0$ (eqn of plane) (implicit eqn for points)

$\boxed{\text{we could stop here but}}$ ← → solve:

$$[-5\ 1\ 1 | 0] \xrightarrow{rref} \begin{array}{cccc} L & F & F & \\ x_1 & x_2 & x_3 & \end{array} [1\ {-1/5}\ {-1/5} | 0]$$

$x_2 = t_1$
$x_3 = t_2$
$x_1 = \frac{1}{5}t_1 + \frac{1}{5}t_2$

$$\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}\frac{1}{5}t_1 + \frac{1}{5}t_2\\t_1\\t_2\end{bmatrix} = t_1 \underbrace{\begin{bmatrix}1/5\\1\\0\end{bmatrix}}_{\vec{u}_1} + t_2 \underbrace{\begin{bmatrix}1/5\\0\\1\end{bmatrix}}_{\vec{u}_2}$$

$\{\vec{u}_1, \vec{u}_2\}$ = new basis

How to express new basis in terms of old basis?

$$a\vec{v}_1 + b\vec{v}_2 = \begin{bmatrix}1&1\\2&3\\3&2\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}1/5\\1\\0\end{bmatrix} = \vec{u}_1$$
$$c\vec{v}_1 + d\vec{v}_2 = \begin{bmatrix}1&1\\2&3\\3&2\end{bmatrix}\begin{bmatrix}c\\d\end{bmatrix} = \begin{bmatrix}1/5\\0\\1\end{bmatrix} = \vec{u}_2$$

solve simultaneously

$$\left[\begin{array}{cc|cc}1&1&1/5&1/5\\2&3&1&0\\3&2&0&1\end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|cc}1&0&-2/5&3/5\\0&1&3/5&-2/5\end{array}\right]$$

$\begin{bmatrix}a\\b\end{bmatrix}$ $\begin{bmatrix}c\\d\end{bmatrix}$

$\therefore B = \frac{1}{5}\begin{bmatrix}-2&3\\3&-2\end{bmatrix}$ basis changing matrix

cols = old coords of new basis

equivalent to: $\begin{bmatrix}1&1\\2&3\\3&2\end{bmatrix}\begin{bmatrix}a&c\\b&d\end{bmatrix} = \begin{bmatrix}1/5&1/5\\1&0\\0&1\end{bmatrix}$

(old) V B (change) U (new) → $VB = U$

any vector in this subspace has unique coords:

$$x_1\vec{v}_1 + x_2\vec{v}_2 = \underbrace{\begin{bmatrix}1&1\\2&3\\3&2\end{bmatrix}}_{\text{old basis}}\underbrace{\begin{bmatrix}x_1\\x_2\end{bmatrix}}_{\text{old coords}} = \underbrace{\begin{bmatrix}1/5&1/5\\1&0\\0&1\end{bmatrix}}_{\text{new basis}}\underbrace{\begin{bmatrix}y_1\\y_2\end{bmatrix}}_{\text{new coords}} = y_1\vec{u}_1 + y_2\vec{u}_2$$

$$V\vec{x} = U\vec{y} = (VB)y = V(B\vec{y})$$

unique

coord transformation

$$\boxed{\begin{aligned}\vec{x} &= B\vec{y}\\ y &= B^{-1}\vec{x}\end{aligned}}$$

$B^{-1} = \frac{1}{5}\begin{bmatrix}-2&3\\3&-2\end{bmatrix}$ goes in the reverse direction