3.1 linear systems of equations: elimination

high school elimination example:

$$\boxed{\begin{aligned} 2x+7y+3z&=11\\ x+3y+2z&=2\\ 3x+7y+9z&=-12\end{aligned}}$$

switch to make elimination arithmetic easier (convenient)

$$\boxed{\begin{aligned} x+3y+2z&=2\\ 2x+7y+3z&=11\\ 3x+7y+9z&=-12\end{aligned}}$$

eliminate x first from last 2 eqns

$$\begin{aligned} 2x+7y+3z&=11\\ *(-2)\quad -2x-6y-4z&=-4\\ \hline y-z&=7\end{aligned}$$

$$\begin{aligned} 3x+7y+9z&=-12\\ *(-3)\quad -3x-9y-6z&=-6\\ \hline -2y+3z&=-18\end{aligned}$$

$$\boxed{\begin{aligned} x+3y+2z&=2\\ y-z&=7\\ -2y+3z&=-18\end{aligned}}$$

eliminate y next from last eqn

$$\begin{aligned} \xrightarrow{*(2)}\quad 2y-2z&=14\\ -2y+3z&=-18\\ \hline z&=-4\end{aligned}$$

$$\boxed{\begin{aligned} x+3y+2z&=2\\ y-z&=7\\ z&=-4\end{aligned}}$$

successive eliminations now complete
(notice "triangular" shape of nonzero terms on LHS)

next backsub from bottom up

$$y=7+(-4)=3$$

$$x=2-3(3)-2(-4) = 2-9+8 = 1$$

$$\boxed{x=1,\ y=3,\ z=-4}$$

unique soln

Interpretation: 3 planes intersect in a single point

These steps will be programmed into a matrix reduction algorithm to solve any linear system of equations

Then we can forget this technique.

"inconsistent" system: 3 planes with no common intersection

see MAPLE

$\boxed{3.1}$ linear systems of equations: elimination

linear DEs are linear for the unknown and its derivatives. Imposing initial conditions leads to linear systems of equations for the arbitrary constants since the general solutions are linear in those constants.

example

IVP: $\underbrace{y''+3y'+2y=0}$, $y(0)=1$, $y'(0)=1$

impose initial conditions ↓

gen soln: $y = c_1 e^{-x} + c_2 e^{-2x}$

$y' = -c_1 e^{-x} - 2c_2 e^{-2x}$

$$\boxed{\begin{aligned} y(0) &= c_1 + c_2 &= 1 \\ y'(0) &= -c_1 - 2c_2 &= 1 \end{aligned}}$$

↓

solve:

$\underbrace{c_1 = 3,\ c_2 = -2}$

never a final result for any calculation: ALWAYS BACKSUB!

$$\boxed{y = 3e^{-x} - 2e^{-2x}}$$

This is "the solution" of the IVP

$\boxed{\text{See Maple}}$

$\boxed{\text{systems of linear DEs}}$ need even more linear mathematics so we have to take a serious detour into enough "linear algebra" to return to systems

The next example is only a PREVIEW of where we are headed to motivate why linear algebra is needed.

PREVIEW: Why LinAlg with DE?

$$\boxed{\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= x \end{aligned}}$$

coupled linear 1st order DEs

($x, y, \frac{dx}{dt}, \frac{dy}{dt}$ appear only linearly, can't solve one DE without solving other)

↓ LinAlg

$$\begin{aligned} +:\quad \frac{dx}{dt} + \frac{dy}{dt} &= y + x \\ -:\quad \frac{dx}{dt} - \frac{dy}{dt} &= y - x \end{aligned} \quad\rightarrow\quad \begin{aligned} \frac{d}{dt}\overbrace{(x+y)}^{u} &= \overbrace{(x+y)}^{u} \\ \frac{d}{dt}\underbrace{(x-y)}_{v} &= -\underbrace{(x-y)}_{v} \end{aligned} \quad\rightarrow\quad \boxed{\begin{aligned} \frac{du}{dt} &= u \\ \frac{dv}{dt} &= -v \end{aligned}}$$

uncoupled linear DEs

solve

$$\rightarrow\ u = c_1 e^{t}$$
$$\rightarrow\ v = c_2 e^{-t}$$

↓

$$\boxed{\begin{aligned} u &= x + y \\ v &= x - y \end{aligned}}$$

linear change of variables

↓ LinAlg

$$\begin{aligned} +:\quad u + v &= 2x \\ -:\quad u - v &= 2y \end{aligned} \quad\rightarrow\quad \boxed{\begin{aligned} x &= \tfrac{1}{2}(u+v) \\ y &= \tfrac{1}{2}(u-v) \end{aligned}}$$

inverse transformation

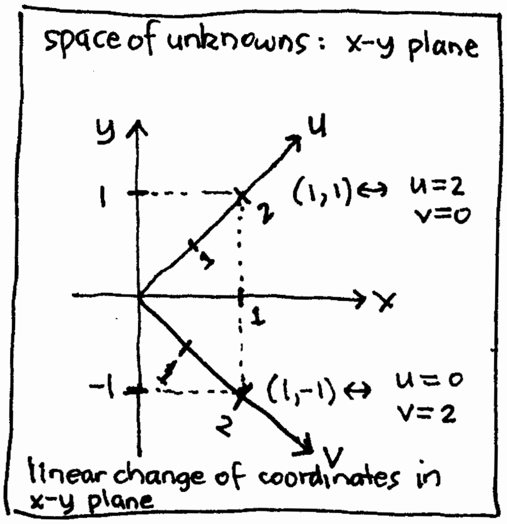
back substitute

$$\boxed{\begin{aligned} x &= \tfrac{1}{2}(c_1 e^{t} + c_2 e^{-t}) \\ y &= \tfrac{1}{2}(c_1 e^{t} - c_2 e^{-t}) \end{aligned}}$$

general solution

geometric interpretation:

space of unknowns: x-y plane



linear change of coordinates in x-y plane

I.C.s ↓

$$\begin{aligned} x(0) &= \boxed{\tfrac{1}{2}(c_1 + c_2) = x_0} \\ y(0) &= \boxed{\tfrac{1}{2}(c_1 - c_2) = y_0} \end{aligned}$$

2x2 linear system of eqns

unknowns ↑↑   given ↑

↓ LinAlg

$$\begin{aligned} +:\quad & \boxed{c_1 = x_0 + y_0} \\ -:\quad & \boxed{c_2 = x_0 - y_0} \end{aligned}$$

soln

back substitute

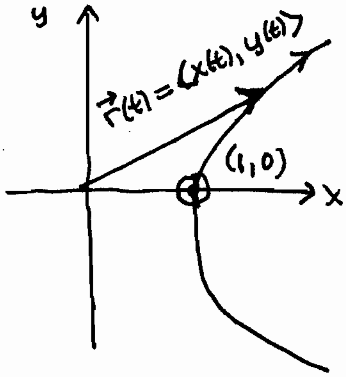
$$\begin{aligned} \text{IVP soln:}\quad x &= \tfrac{1}{2}\left[(x_0+y_0)e^{t} + (x_0-y_0)e^{-t}\right] = x_0\left(\frac{e^{t}+e^{-t}}{2}\right) + y_0\left(\frac{e^{t}-e^{-t}}{2}\right) \\ y &= \tfrac{1}{2}\left[(x_0+y_0)e^{t} - (x_0-y_0)e^{-t}\right] \underset{\substack{\uparrow \\ \text{regroup} \\ x_0, y_0 \text{ terms}}}{=} x_0\underbrace{\left(\frac{e^{t}-e^{-t}}{2}\right)}_{\sinh t} + y_0\underbrace{\left(\frac{e^{t}+e^{-t}}{2}\right)}_{\cosh t} \end{aligned}$$

linear combinations of exponentials

LinAlg operations:
addition, subtraction, multiplication by a constant

PREVIEW: Why Lin Alg with DE? (2)

interpretation



Suppose we consider the initial data point $(x_0, y_0) = (1,0)$ in the x-y plane.

Then the solution represents a parametrized curve in the x-y plane whose position vector is

$$\vec r(t) = \langle x(t), y(t)\rangle = \langle \cosh t, \sinh t\rangle = \left\langle \frac{e^t+e^{-t}}{2}, \frac{e^t-e^{-t}}{2}\right\rangle$$

and tangent vector is

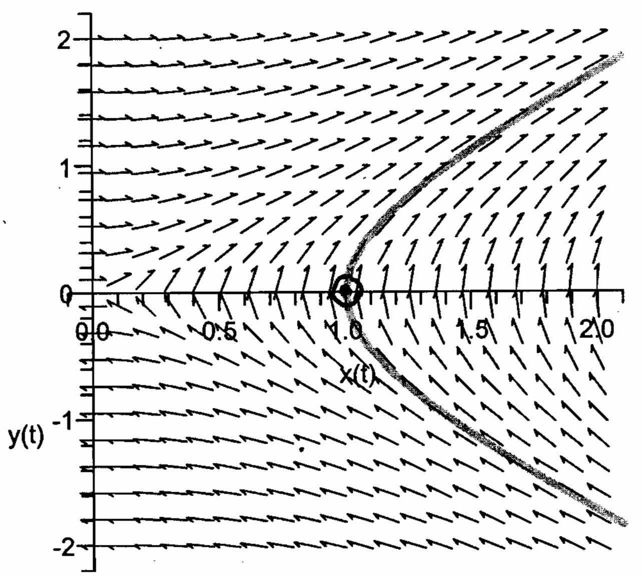
$$\vec r\,'(t) = \langle x'(t), y'(t)\rangle = \langle \sinh t, \cosh t\rangle = \left\langle \frac{e^t-e^{-t}}{2}, \frac{e^t+e^{-t}}{2}\right\rangle$$

in vector form the DE system is just

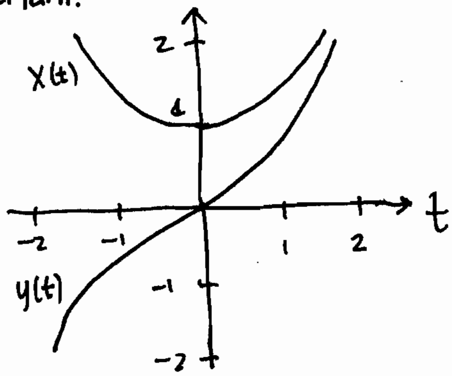
$$\vec r\,'(t) = \langle y(t), x(t)\rangle$$

so the tangent to the solution curve equals the value of the vector field $\vec F(x,y) = \langle y, x\rangle$ at each point along that curve.

Thus plotting the directionfield for this vector field gives us a picture of the family of solution curves which connect up the arrows.



Of course we are also interested in separate plots of x or y versus t in applications where the individual behavior of x(t) and y(t) is usually important.



Remark: $x^2 - y^2 = \cosh^2 t - \sinh^2 t = \left(\frac{e^t+e^{-t}}{2}\right)^2 - \left(\frac{e^t-e^{-t}}{2}\right)^2 = \frac{e^{2t}+2+e^{-2t}-(e^{2t}-2+e^{-2t})}{4}$
$= 1$!

This particular solution is the hyperbolic analog of the usual angle parametrized unit circle, namely a hyperbola. Hyperbolic geometry turns out to be important in special relativity.