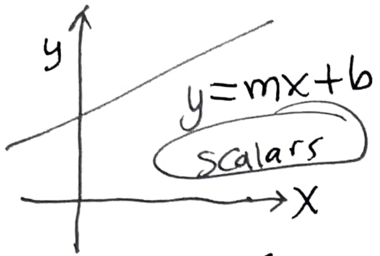
vector field calculus

how did we get here, where are we going?

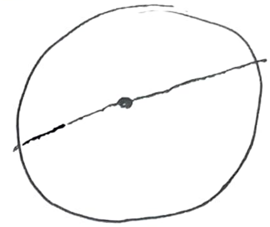
linear stuff



→ calc 1

$y = f(x) \rightarrow \frac{dy}{dx}$

zoom in till linear



define derivative as in linear case

↓

calc 2 integration $\int_a^b f(x)\,dx$

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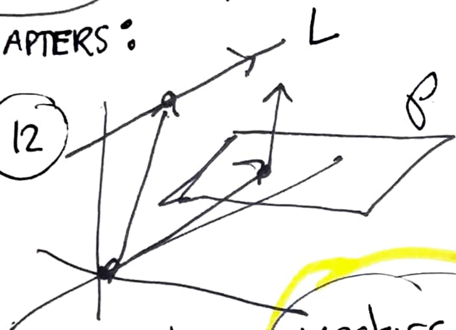


lines, planes, vectors!

→ 13 vector dep. var. $\vec{F}(t) = \langle F_1(t), F_2(t), F_3(t) \rangle$

one ind. var.

"parametrized curves" to visualize

$\vec{r} = \langle x, y, z \rangle = \vec{F}(t)$

↓ $\vec{F}'(t)$, $\vec{F}''(t)$ derivatives

$\int_a^b F(t)\,dt$ integrals

14 vector ind. var.: one dep. var

$z = f(x, y) = f(\vec{r})$ 2-d

$w = F(x, y, z) = F(\vec{r})$ 3-d

diff

↓ diff $\vec{\nabla} F(\vec{r})$ gradient = vector derivative.

15 vector ind. var. one dep. var

int

$\iint_R F(\vec{r})\,dA$, $\iiint_R F(\vec{r})\,dV$

16 vector ind. var. + vector dep. var. = "vector field"

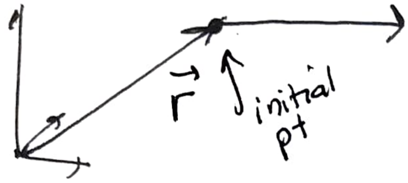
$\vec{F}(\vec{r}) = \langle F_1(\vec{r}), F_2(\vec{r}) \rangle$ or $\langle F_1(\vec{r}), F_2(\vec{r}), F_3(\vec{r}) \rangle$

↓

diff: div, curl, grad

int: "line" & surface integrals (not in syllabus)

vector field calculus (2) visualization : sample on a grid



$\vec{F}(\vec{r})$ ← defined at each pt of space (2-d or 3-d)

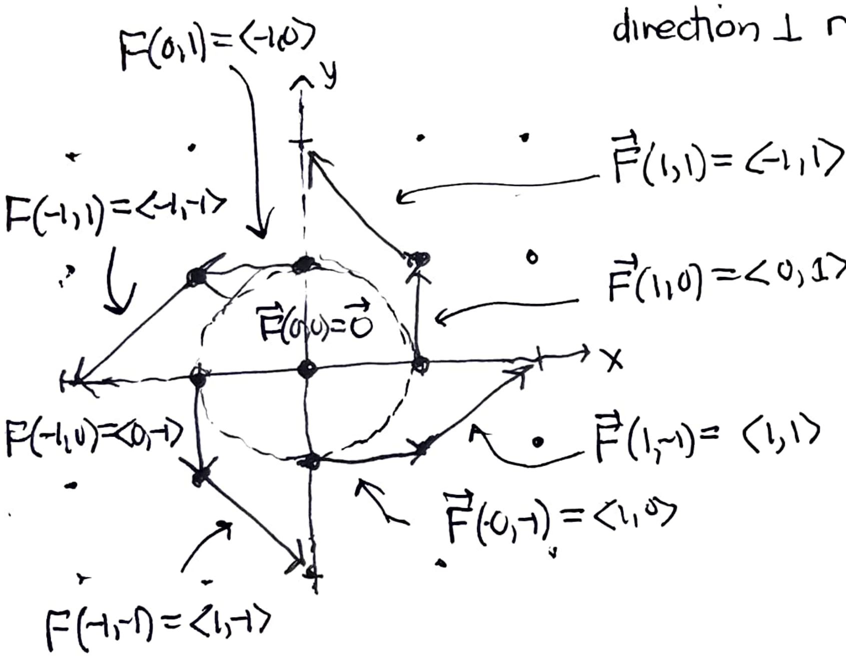
2-d example

$\vec{F}(x,y) = \langle -y, x \rangle$ → note $|\vec{F}(x,y)| = \sqrt{x^2+y^2} = r$

length same on each circle

$\vec{r} \cdot \vec{F}(x,y) = \langle x,y \rangle \cdot \langle -y,x \rangle = -xy + xy = 0$

direction $\perp$ radial direction at each pt.



technology is better at this

must "resize" vectors to fit into grid boxes

to avoid overlapping chaos

vector field calculus (3)

gradient vector fields (very special)

$$f(\vec{r}) \longrightarrow \underbrace{\vec{\nabla} f(\vec{r})}_{\text{orthogonal to level curves, surfaces}} = \vec{F}(r)$$

example: inverse square force field

$$f = \frac{k}{\sqrt{x^2+y^2+z^2}} = \frac{k}{|\vec{r}|} \quad \left(= \frac{k}{\rho}\right)$$

$$= k\,(x^2+y^2+z^2)^{-1/2}$$

$\frac{\partial}{\partial x}(x^2+y^2+z^2)$ chain rule

$$\frac{\partial f}{\partial x} = k\left(-\frac{1}{2}\right)(x^2+y^2+z^2)^{-3/2} \cdot (2x)$$

(scalar derivatives more useful as a vector derivative)

$$\frac{\partial f}{\partial y} = \quad \ldots\ldots \quad (2y)$$

$$\frac{\partial f}{\partial z} = \quad \ldots\ldots \quad (2z)$$

$$\vec{F} = \vec{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = -\frac{k\,\langle x, y, z\rangle}{(x^2+y^2+z^2)^{3/2}} = -\frac{k\,\vec{r}}{|\vec{r}|^3} = -\frac{k\,\hat{r}}{|\vec{r}|^2}$$

points radially inward ($k>0$) or outward ($k<0$)

inverse square distance from origin

level surfaces are concentric spheres around origin

$$|\vec{F}| = \frac{|k|}{|\vec{r}|^2}$$ magnitude is inverse square distance proportional