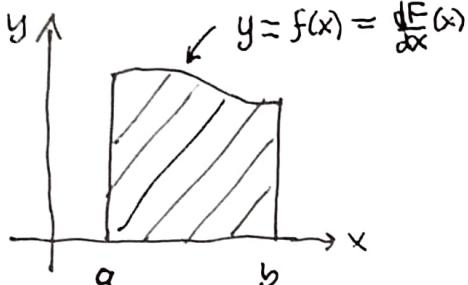


Higher Dimensional Generalization of Fundamental Theorem of Calculus

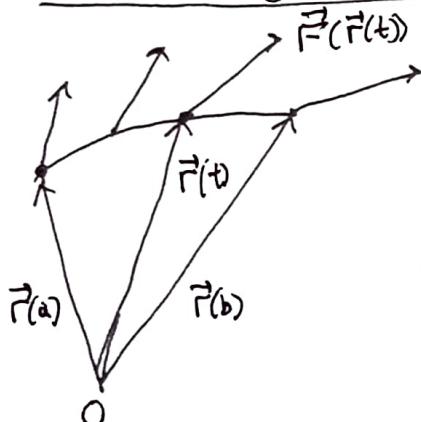
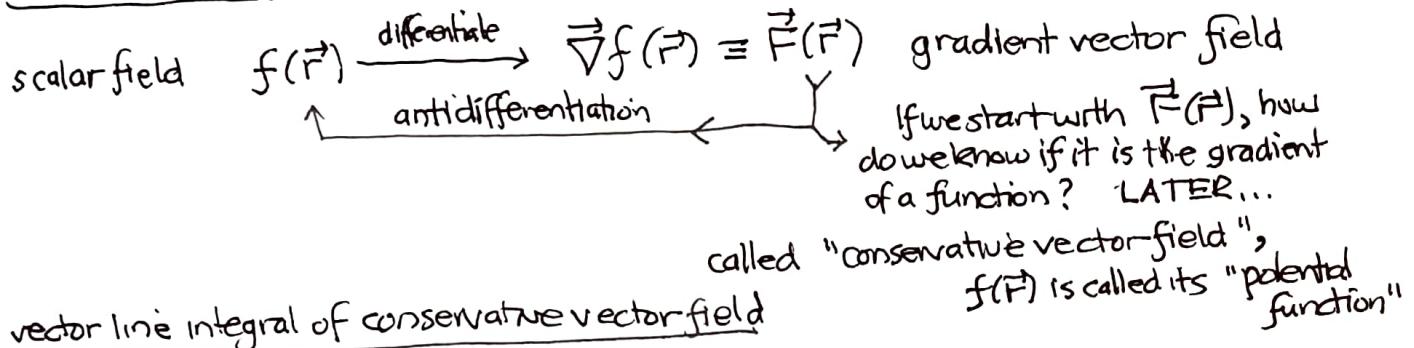


Fundamental Theorem of Calculus:

$$\int_a^b f(x) dx = \int_a^b \frac{dF(x)}{dx} dx = F(x) \Big|_a^b = F(b) - F(a)$$

definite integrals are evaluated using antiderivation

HIGHER DIMENSION



$$C: \vec{r} = \vec{r}(t), t = a..b$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt = \int_a^b \frac{df(\vec{r}(t))}{dt} dt = f(\vec{r}(t)) \Big|_a^b = f(\vec{r}(b)) - f(\vec{r}(a))$$

final - initial

$$\vec{F} \cdot d\vec{r} = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

$= df$ is an "exact" differential

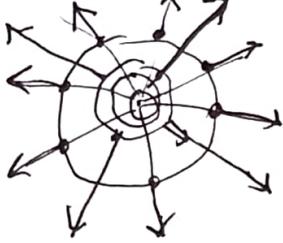
The line integral of a conservative vector field is just the difference in its potential function between the initial and terminal point.

Higher Dim Fund Theorem & Conservative Vector Fields (2)

Example : inverse square force field

potential function $f = \frac{k}{|\vec{r}|} \rightarrow \vec{F} = \vec{\nabla}f = -\frac{k\hat{r}}{|\vec{r}|^2}$ force field

level curves/surfaces are concentric circles / spheres about origin in 2-d / 3-d



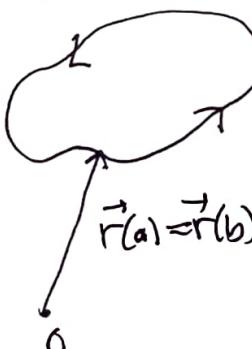
points in radial direction towards ($k > 0$) away from ($k < 0$) the origin

undefined at origin!
domain omits origin!
(location of point charge or mass)

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt = f(\vec{r}(b)) - f(\vec{r}(a)) = \Delta f_{\vec{r}}|_a^b$$

$$= \frac{k}{|\vec{r}(b)|} - \frac{k}{|\vec{r}(a)|} \quad \text{independent of "path"}$$

conservative vector fields have path independent line integrals



on a simple closed curve (starting pt = ending pt)

$$\int_C \vec{F} \cdot d\vec{r} = 0$$

"O" $\int_C \vec{F} \cdot d\vec{r}$ ← line integral around closed loop
 "simple" means no self-intersections

conservative vector fields have vanishing line integral around any simple closed curve.

is the converse statement true? It depends on the details

Higher Dim Fund Theorem & Conservative Vector Fields (3)

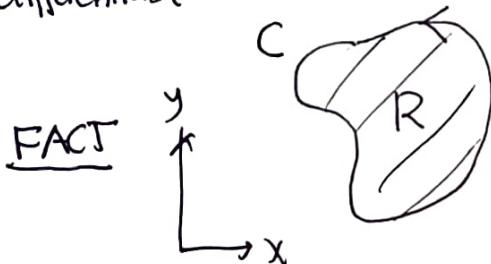
2-d case: $\vec{F} = \vec{\nabla}f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle F_1, F_2 \rangle$

$\left. \begin{array}{l} \frac{\partial f}{\partial x} = F_1 \\ \frac{\partial f}{\partial y} = F_2 \end{array} \right\}$ If we start with \vec{F} and want to "antidifferentiate" to go back to f which we don't know if it exists, we must solve this pair of partial differential equations for the unknown solution function $f(x, y)$

↓ consequence (necessary)

$$\left. \begin{array}{l} \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial F_1}{\partial y} \\ \text{or} \\ \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial F_2}{\partial x} \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0 \end{array} \right.$$

order does not matter if f differentiable



FACT

Suppose \vec{F} satisfies this condition on a region of ~~space~~ the plane (2-d case), does that guarantee a solution f exists?

If path C encloses a "simply connected" (no holes) region, where this condition is satisfied everywhere, then \vec{F} does have a potential f . But how to find?

2 methods exist to find f .

- a) successive line integrals from reference point
- b) solve PDEs

Higher Dim Fund Theorem & Conservative Vector Fields (4)

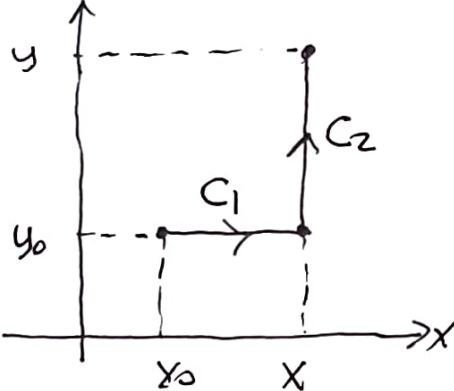
a) successive line integration

1-d case: $\int f(x) dx = \underbrace{F(x)}_{\text{"constant"}} + C$

Define $F(x) = \int_a^x f(u) du \rightarrow F'(x) = f(x) !$

integral formula for an antiderivative
if can find way to evaluate this
formula, it gives us a particular
antiderivative

2-d case:



$C_1: y = y_0 \text{ while } x = x_0 \dots x$

$$\vec{r} = \langle t, y_0 \rangle, t = x_0 \dots x \quad (dy=0)$$

$C_2: "x=x" \text{ while } y = y_0 \dots y$

$$\vec{r} = \langle x, t \rangle, t = y_0 \dots y \quad (dx=0)$$

$$\vec{r}' = \langle 1, \vec{0} \rangle \quad \vec{F} \cdot \vec{r}' = F_1$$

$$\vec{r}' = \langle 0, 1 \rangle \quad \vec{F} \cdot \vec{r}' = F_2$$

since line integral should not depend on the path,
choose this path from (x_0, y_0) to (x, y) :

$$f(\vec{F}) - \underbrace{f(\vec{F}_0)}_C = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_{C_1} F_1 dx + \int_{C_2} F_2 dy$$

$$= \int_{x_0}^x F(t, y_0) dt + \int_{y_0}^y F(x, t) dt$$

yeah, sure, in principle
but easier by method b)!

b) solve PDE's: solve one by partial integration, plug result into other, then solve second equation.

example. $\vec{F} = \langle 3+2xy, x^2-3y^2 \rangle \rightarrow \frac{\partial}{\partial x}(x^2-3y^2) - \frac{\partial}{\partial y}(3+2xy) = 2x-2x = 0 \checkmark$

$$\frac{\partial f}{\partial x} = 3+2xy \xrightarrow{\int dx} \int \frac{\partial f}{\partial x} dx = \int 3+2xy dx = 3x+x^2y + \underbrace{C(y)}_{\text{"constant" of integration}}$$

$$\frac{\partial f}{\partial y} = x^2-3y^2 \xleftarrow{\text{"f}} \text{so } f = 3x+x^2y + C(y)$$

$$\boxed{\begin{aligned} \frac{\partial}{\partial y}(3x+x^2y + C(y)) &= x^2-3y^2 \\ &= 0 + x^2 + C'(y) \end{aligned}}$$

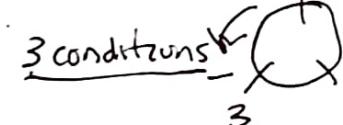
$$\left. \begin{aligned} C'(y) &= -3y^2 \quad (\text{no } x!) \\ C(y) &= \int -3y^2 dy = -y^3 + k \end{aligned} \right\} \text{so } f = 3x+x^2y - y^3 + k \cdot \text{done.}$$

Higher Dim Fund Theorem & Conservative Vector Fields (5)

3-d case

$$\vec{F} = \langle F_1, F_2, F_3 \rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

$$\begin{aligned} \frac{\partial f}{\partial x} = F_1 &\quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial F_1}{\partial y} \quad \frac{\partial F_2 - \partial F_1}{\partial x - \partial y} = 0 \quad \leftarrow \text{2-d condition if } F_3 \equiv 0 \\ \frac{\partial f}{\partial y} = F_2 &\quad \frac{\partial^2 f}{\partial z \partial x} = \frac{\partial F_1}{\partial z} - \frac{\partial F_2}{\partial x} \\ \frac{\partial f}{\partial z} = F_3 &\quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial F_2}{\partial x} - \frac{\partial F_3}{\partial y} \\ &\quad \frac{\partial^2 f}{\partial x \partial z} = \frac{\partial F_3}{\partial x} \quad \frac{\partial F_3 - \partial F_2}{\partial y - \partial z} = 0 \\ &\quad \underbrace{\frac{\partial^2 f}{\partial y \partial z}}_{\text{equal in pairs!}} = \frac{\partial F_3}{\partial y} \end{aligned}$$

3 conditions  + $\frac{\partial F_2 - \partial F_1}{\partial x - \partial z}$
 clock
anticlock
 where $(x_1, x_2, x_3) = (x, y, z)$
 permute $(1, 2, 3)$

Example

$$\vec{F} = \langle e^{xz}yz, e^{xz}, e^{xz}xy \rangle = \langle F_1, F_2, F_3 \rangle$$

$$\left\{ \begin{array}{l} \frac{\partial F_2 - \partial F_1}{\partial x - \partial y} = \frac{\partial}{\partial x}(e^{xz}) - \frac{\partial}{\partial y}(e^{xz}yz) = ze^{xz} - ze^{xz} = 0 \checkmark \\ \frac{\partial F_3 - \partial F_2}{\partial y - \partial z} = \frac{\partial}{\partial y}(e^{xz}xy) - \frac{\partial}{\partial z}(e^{xz}) = xe^{xz} - xe^{xz} = 0 \checkmark \\ \frac{\partial F_1 - \partial F_3}{\partial z - \partial x} = \frac{\partial}{\partial z}(e^{xz}yz) - \frac{\partial}{\partial x}(e^{xz}xy) = \cancel{xyz}e^{xz} + \cancel{ye^{xz}} - (\cancel{xyz}e^{xz} + \cancel{ye^{xz}}) = 0 \checkmark \end{array} \right.$$

guarantees soln exists. solve 3 PDEs in any order.

$$\int \left[\frac{\partial f}{\partial x} = yz e^{xz} \right] dx \rightarrow f = \int yz e^{xz} dx = yz \cancel{e^{xz}} + C(y, z) = ye^{xz} + C(y, z)$$

$\frac{\partial f}{\partial y} = e^{xz} \quad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (ye^{xz} + C(y, z)) = e^{xz} + \frac{\partial C}{\partial y} (y, z)$

$0 = \frac{\partial C}{\partial y} (y, z) \xrightarrow{\text{ind of } y} C(y, z) = C(z)$

$\frac{\partial f}{\partial z} = xy e^{xz} \quad \text{so } f = ye^{xz} + C(z)$

$\frac{\partial f}{\partial z} = xy e^{xz} + C'(z) \quad C' = C'(z) \xrightarrow{\text{ind of } z} C(z) = k$

if terms had not cancelled,
successive eqns would have
been inconsistent!

$$\text{so } f = ye^{xz} + k \quad \checkmark$$

Higher Dim Fund Theorem & Conservative Vector Fields (6)

Counter-example

$$\vec{F} = \langle F_1, F_2, F_3 \rangle = \langle y, z, x \rangle$$

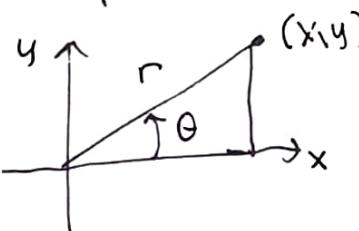
$$\int \left[\frac{\partial f}{\partial x} = y \right] dx \rightarrow f = \int y dx = xy + C(y, z)$$

$$\frac{\partial f}{\partial y} = z \quad \frac{\partial f}{\partial y} = x + C(y, z) \rightarrow z = x + C(y, z)$$

$$\frac{\partial f}{\partial z} = x$$

$C(y, z) = z - x$
 ↑
 no x on LHS
 makes no sense,
 inconsistent eqns
 no soln.

example



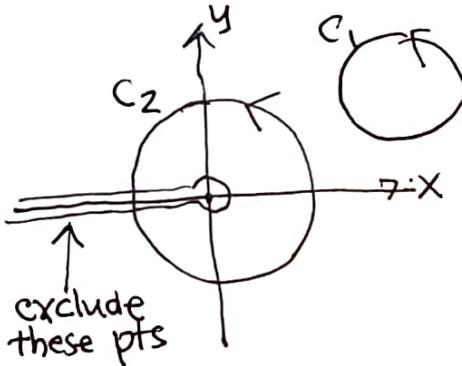
$$\tan \theta = \frac{y}{x} \rightarrow \theta = \arctan(y/x) = \begin{cases} \arctan(y/x), & x > 0 \text{ (quads 1, 4)} \\ \arctan(y/x) + \pi, & x < 0, y > 0 \text{ (quad 2)} \\ \arctan(y/x) - \pi, & x < 0, y < 0 \text{ (quad 3)} \\ \frac{\pi}{2}, & x = 0, y > 0 \\ -\frac{\pi}{2}, & x = 0, y < 0 \end{cases}$$

undefined at $x=0=y$
 continuous and differentiable
 everywhere except on
 negative x -axis —
 jump discontinuity: $\Delta\theta = 2\pi$!

$$d\theta = \frac{\partial \theta}{\partial x} dx + \frac{\partial \theta}{\partial y} dy = \frac{1}{1+(y/x)^2} \left(-\frac{y}{x^2} \right) dx + \frac{1}{1+(y/x)^2} \left(\frac{1}{x} \right) dy$$

$$= \frac{-y dx + x dy}{x^2 + y^2} = \underbrace{\langle -y, x \rangle}_{x^2 + y^2} \cdot \langle dx, dy \rangle$$

$\vec{F} \equiv \vec{\nabla} \theta$ so $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$ since \vec{F} is a gradient



For any loop that does not cross "bad points"
 the line integral vanishes

But any loop passing thru "bad points"
 has nonzero value $2\pi = \Delta\theta$ if in
 counterclockwise direction

$$\oint_{C_1} d\theta = 0, \quad \oint_{C_2} d\theta = \Delta\theta = 2\pi$$