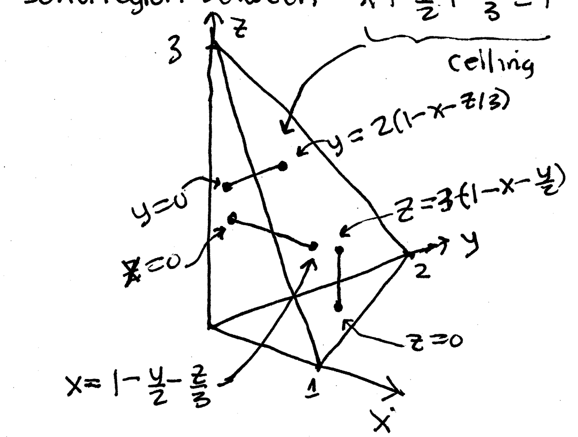
Examples of iterating triple integrals

Solid region between $\underbrace{x + \frac{y}{2} + \frac{z}{3} = 1}_{\text{ceiling}}$ and $\underbrace{x=0,\ y=0}_{\text{side walls}}$, $\underbrace{z=0}_{\text{floor}}$ planes. Find Volume.

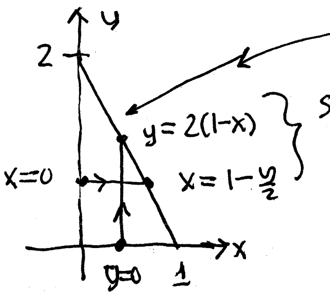


$\boxed{\text{innermost integral limits}}$ come from starting and stopping values of variable along cross-section line segments.

$\boxed{\text{outer double integral limits}}$ come from projections of solid regions onto 3 coordinate planes, only need 2d diagrams

z first: $z = 0 \,..\, 3(1-x-\frac{y}{2})$

$x + \frac{y}{2} + \frac{z}{3} = 1$ intersects $z=0$ at $x + \frac{y}{2} = 1$



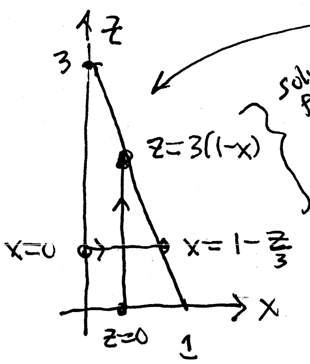
solved for each variable

$$\int_0^1 \int_0^{2(1-x)} \int_0^{3(1-x-\frac{y}{2})} 1 \, dz\,dy\,dx$$

$$\int_0^2 \int_0^{1-\frac{y}{2}} \int_0^{3(1-x-\frac{y}{2})} 1 \, dz\,dx\,dy$$

y first: $y = 0 \,..\, 2(1-x-\frac{z}{3})$

$x + \frac{y}{2} + \frac{z}{3} = 1$ intersects $y=0$ at $x + \frac{z}{3} = 1$



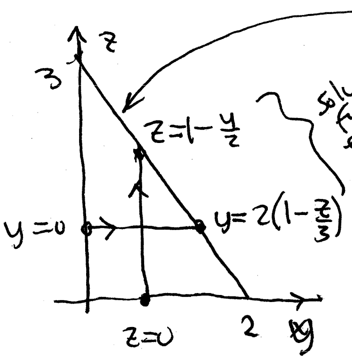
solved for each variable

$$\int_0^1 \int_0^{3(1-x)} \int_0^{2(1-x-z/3)} 1 \, dy\,dz\,dx$$

$$\int_0^3 \int_0^{1-z/3} \int_0^{2(1-x-z/3)} 1 \, dy\,dx\,dz$$

x first: $x = 0 \,..\, 1 - y/2 - z/3$

$x + \frac{y}{2} + \frac{z}{3} = 1$ intersects $x=0$ at $\frac{y}{2} + \frac{z}{3} = 1$



solved for each variable

$$\int_0^3 \int_0^{2(1-z/3)} \int_0^{1-y/2-z/3} 1 \, dx\,dy\,dz$$

$$\int_0^2 \int_0^{1-y/2} \int_0^{1-y/2-z/3} 1 \, dx\,dz\,dy$$

all six integrals give same result.

deconstructing a triple integral & using rotational symmetry

$$\int_{-2}^{2}\int_{x^2}^{4}\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\, dz\, dy\, dx$$

sqrts are bad news for integrals☹
Maple cannot evaluate this. What to do?

integration leads to more complication, stops Maple after 1st integration

$$\int_{x=-2}^{x=2}\int_{y=x^2}^{y=4}\int_{z=-\sqrt{y-x^2}}^{z=\sqrt{y-x^2}} \sqrt{x^2+z^2}\, dz\, dy$$

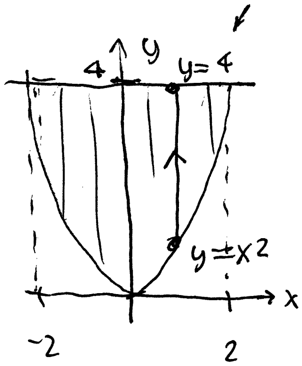
← annotate limits of integration to get equations of surfaces and lines bounding triple & double integration regions

easy 2-d diagram

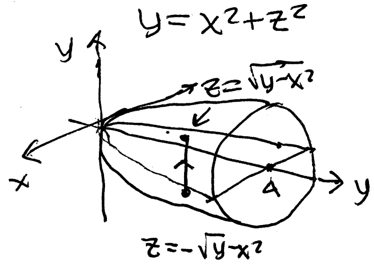
$z=\pm\sqrt{y-x^2}$ upper lower graphs

$z^2=y-x^2$

$y=x^2+z^2$ parabola of revolution about y axis. cut off by plane $y=4$ from left diagram



outer double integral



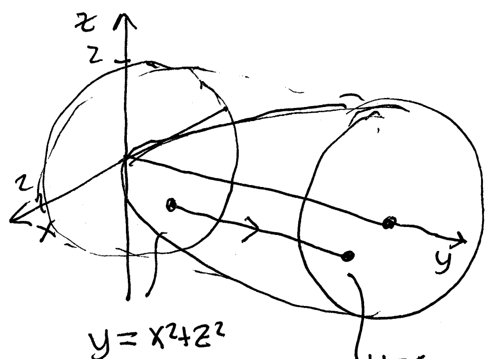
innermost integral

in fact $z=0$ cross-section is exactly the left diagram which separates upper & lower graphs

innermost integral goes from bottom to top

To take advantage of rotational symmetry we should integrate first in y direction.

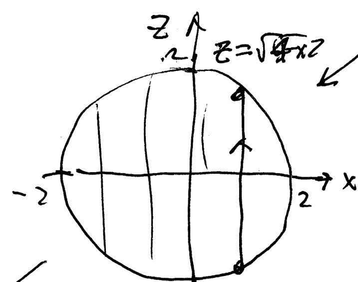
$y=x^2+z^2$ intersects $y=4$ at $x^2+z^2=4$
circle of radius 2 in xz plane
(projection onto xz plane)



$y=x^2+z^2$

$y=4$

innermost integral



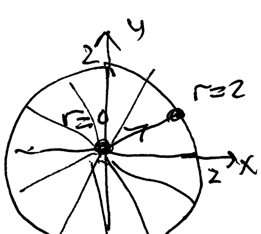
outer double integral

But Maple cannot do outer double integral in this order either (stops after 2nd integration)

If we keep Cartesian coords:

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\, dy\, dz\, dx$$

$dy\,(r\,dr\,d\theta)$



$\theta=0..2\pi$

$x=r\cos\theta$
$z=r\sin\theta$
$x^2+z^2=r^2$
introduce polar coords in xz plane

$$=\int_0^{2\pi}\int_0^{2}\int_{r^2}^{4} r\ r\,dy\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^{2}\int_{r^2}^{4} r^2\,dy\,dr\,d\theta$$

$r^2 y\big|_{r^2}^{4} = r^2(4-r^2)$

$$=\underbrace{\int_0^{2\pi} d\theta}_{2\pi}\ \int_0^2 4r^2-r^4\,dr = \frac{2^7}{15}\pi = \boxed{\frac{128}{15}\pi}$$

$\frac{4r^3}{3}-\frac{r^5}{5}\Big|_0^2 = \frac{2^5}{3}-\frac{2^5}{5} = 2^5\left(\frac{2}{15}\right)=\frac{2^6}{15}$

3d diagram tells you starting & stopping values of innermost integral.
projection onto coord plane gives outer double integral.