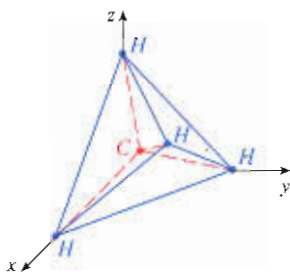$(0, 0, 1)$, and $(1, 1, 1)$ as shown in the figure. Then the centroid is $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.]

54. If $c = |a|b + |b|a$, where **a**, **b**. and **c** are all nonzero vectors, show that **c** bisects the angle between **a** and **b**.

55. Prove Properties 2, 4, and 5 of the dot product (Theorem 2).

56. Suppose that all sides of a quadrilateral are equal in length and opposite sides are parallel. Use vector methods to show that the diagonals are perpendicular.

57. Use Theorem 3 to prove the *Cauchy-Schwarz Inequality:*

$$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$$

58. The *Triangle Inequality for vectors* is

$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$$

(a) Give a geometric interpretation of the Triangle Inequality.
(b) Use the Cauchy-Schwarz Inequality from Exercise 57 to prove the Triangle Inequality.   [*Hint:* Use the fact that $|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$ and use Property 3 of the dot product.]

59. The Parallelogram Law states that

$$|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2$$

(a) Give a geometric interpretation of the Parallelogram Law.
(b) Prove the Parallelogram Law. (See the hint in Exercise 58.)

||| **12.4** The Cross Product

The **cross product** $\mathbf{a} \times \mathbf{b}$ of two vectors **a** and **b**, unlike the dot product, is a vector. For this reason it is also called the **vector product**. Note that $\mathbf{a} \times \mathbf{b}$ is defined only when **a** and **b** are *three-dimensional* vectors.

> **1 Definition** If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of **a** and **b** is the vector
>
> $$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

This may seem like a strange way of defining a product. The reason for the particular form of Definition 1 is that the cross product defined in this way has many useful properties, as we will soon see. In particular, we will show that the vector $\mathbf{a} \times \mathbf{b}$ is perpendicular to both **a** and **b**.

In order to make Definition 1 easier to remember, we use the notation of determinants. A **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For example,

$$\begin{vmatrix} 2 & 1 \\ -6 & 4 \end{vmatrix} = 2(4) - 1(-6) = 14$$

A **determinant of order 3** can be defined in terms of second-order determinants as follows:

$$\boxed{2} \qquad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Observe that each term on the right side of Equation 2 involves a number a_i in the first row of the determinant, and a_i is multiplied by the second-order determinant obtained from the left side by deleting the row and column in which a_i appears. Notice also the minus sign in the second term. For example,

$$\begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ -5 & 4 & 2 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ 4 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ -5 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 0 \\ -5 & 4 \end{vmatrix}$$

$$= 1(0 - 4) - 2(6 + 5) + (-1)(12 - 0) = -38$$

If we now rewrite Definition 1 using second-order determinants and the standard basis vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. we see that the cross product of the vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ is

$$\boxed{3} \qquad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$$

In view of the similarity between Equations 2 and 3, we often write

$$\boxed{4} \qquad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Although the first row of the symbolic determinant in Equation 4 consists of vectors, if we expand it as if it were an ordinary determinant using the rule in Equation 2, we obtain Equation 3. The symbolic formula in Equation 4 is probably the easiest way of remembering and computing cross products.

EXAMPLE 1 If $\mathbf{a} = \langle 1, 3, 4 \rangle$ and $\mathbf{b} = \langle 2, 7, -5 \rangle$, then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix}\mathbf{k}$$

$$= (-15 - 28)\mathbf{i} - (-5 - 8)\mathbf{j} + (7 - 6)\mathbf{k} = -43\mathbf{i} + 13\mathbf{j} + \mathbf{k}$$

EXAMPLE 2 Show that $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for any vector $\mathbf{a}$ in V_3.

SOLUTION If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, then

$$\mathbf{a} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= (a_2 a_3 - a_3 a_2)\mathbf{i} - (a_1 a_3 - a_3 a_1)\mathbf{j} + (a_1 a_2 - a_2 a_1)\mathbf{k}$$

$$= 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$$

One of the most important properties of the cross product is given by the following theorem.

> **5 Theorem** The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

Proof In order to show that $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$, we compute their dot product as follows:

$$
\begin{aligned}
(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3 \\
&= a_1(a_2 b_3 - a_3 b_2) - a_2(a_1 b_3 - a_3 b_1) + a_3(a_1 b_2 - a_2 b_1) \\
&= a_1 a_2 b_3 - a_1 b_2 a_3 - a_1 a_2 b_3 + b_1 a_2 a_3 + a_1 b_2 a_3 - b_1 a_2 a_3 \\
&= 0
\end{aligned}
$$

A similar computation shows that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$. Therefore, $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. ∎
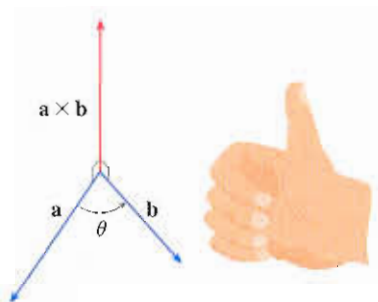
If $\mathbf{a}$ and $\mathbf{b}$ are represented by directed line segments with the same initial point (as in Figure 1), then Theorem 5 says that the cross product $\mathbf{a} \times \mathbf{b}$ points in a direction perpendicular to the plane through $\mathbf{a}$ and $\mathbf{b}$. It turns out that the direction of $\mathbf{a} \times \mathbf{b}$ is given by the *right-hand rule:* If the fingers of your right hand curl in the direction of a rotation (through an angle less than 180°) from $\mathbf{a}$ to $\mathbf{b}$, then your thumb points in the direction of $\mathbf{a} \times \mathbf{b}$.

Now that we know the direction of the vector $\mathbf{a} \times \mathbf{b}$, the remaining thing we need to complete its geometric description is its length $|\mathbf{a} \times \mathbf{b}|$. This is given by the following theorem.

> **6 Theorem** If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (so $0 \leq \theta \leq \pi$), then
> $$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \text{ siu } \theta$$

FIGURE 1

Visual 12 4 shows how $\mathbf{a} \times \mathbf{b}$ changes as $\mathbf{b}$ changes

Proof From the definitions of the cross product and length of a vector, we have

$$
\begin{aligned}
|\mathbf{a} \times \mathbf{b}|^2 &= (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2 \\
&= a_2^2 b_3^2 - 2a_2 a_3 b_2 b_3 + a_3^2 b_2^2 + a_3^2 b_1^2 - 2a_1 a_3 b_1 b_3 + a_1^2 b_3^2 \\
&\quad + a_1^2 b_2^2 - 2a_1 a_2 b_1 b_2 + a_2^2 b_1^2 \\
&= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2 \\
&= |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \\
&= |\mathbf{a}|^2 |\mathbf{b}|^2 - |\mathbf{a}|^2 |\mathbf{b}|^2 \cos^2 \theta \qquad \text{(by Theorem 12.3.3)} \\
&= |\mathbf{a}|^2 |\mathbf{b}|^2 (1 - \cos^2 \theta) \\
&= |\mathbf{a}|^2 |\mathbf{b}|^2 \sin^2 \theta
\end{aligned}
$$

Taking square roots and observing that $\sqrt{\sin^2 \theta} = \text{siu } \theta$ because $\sin \theta \geq 0$ when $0 \leq \theta \leq \pi$, we have

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$ ∎

Geometric characterization of $\mathbf{a} \times \mathbf{b}$

Since a vector is completely determined by its magnitude and directiou, we can now say that $\mathbf{a} \times \mathbf{b}$ is the vector that is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, whose orientation is deter-

mined by the right-hand rule, and whose length is $|\mathbf{a}|\,|\mathbf{b}|\sin\theta$. In fact, that is exactly how physicists *define* $\mathbf{a}\times\mathbf{b}$.

> **[7] Corollary** Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if
>
> $$\mathbf{a}\times\mathbf{b}=\mathbf{0}$$

Proof Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $\theta=0$ or π. In either case $\sin\theta=0$, so $|\mathbf{a}\times\mathbf{b}|=0$ and therefore $\mathbf{a}\times\mathbf{b}=\mathbf{0}$.

The geometric interpretation of Theorem 6 can be seen by looking at Figure 2. If $\mathbf{a}$ and $\mathbf{b}$ are represented by directed liue segments with the same initial point, then they determine a parallelogram with base $|\mathbf{a}|$, altitude $|\mathbf{b}|\sin\theta$, and area

$$A=|\mathbf{a}|(|\mathbf{b}|\sin\theta)=|\mathbf{a}\times\mathbf{b}|$$

FIGURE 2

Thus, we have the following way of interpreting the magnitude of a cross product.

> The length of the cross product $\mathbf{a}\times\mathbf{b}$ is equal to the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.

EXAMPLE 3 Find a vector perpendicular to the plane that passes through the points $P(1,4,6)$, $Q(-2,5,-1)$, and $R(1,-1,1)$.

SOLUTION The vector $\overrightarrow{PQ}\times\overrightarrow{PR}$ is perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ and is therefore perpendicular to the plane through P, Q, and R. We know from (12.2.1) that

$$\overrightarrow{PQ}=(-2-1)\mathbf{i}+(5-4)\mathbf{j}+(-1-6)\mathbf{k}=-3\mathbf{i}+\mathbf{j}-7\mathbf{k}$$
$$\overrightarrow{PR}=(1-1)\mathbf{i}+(-1-4)\mathbf{j}+(1-6)\mathbf{k}=-5\mathbf{j}-5\mathbf{k}$$

We compute the cross product of these vectors:

$$\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-3&1&-7\\0&-5&-5\end{vmatrix}$$

$$=(-5-35)\mathbf{i}-(15-0)\mathbf{j}+(15-0)\mathbf{k}=-40\mathbf{i}-15\mathbf{j}+15\mathbf{k}$$

So the vector $\langle-40,-15,15\rangle$ is perpendicular to the given plane. Any nonzero scalar multiple of this vector, such as $\langle-8,-3,3\rangle$, is also perpendicular to the plane.

EXAMPLE 4 Find the area of the triangle with vertices $P(1,4,6)$, $Q(-2,5,-1)$, and $R(1,-1,1)$.

SOLUTION In Example 3 we computed that $\overrightarrow{PQ}\times\overrightarrow{PR}=\langle-40,-15,15\rangle$. The area of the parallelogram with adjacent sides PQ and PR is the leugth of this cross product:

$$|\overrightarrow{PQ}\times\overrightarrow{PR}|=\sqrt{(-40)^2+(-15)^2+15^2}=5\sqrt{82}$$

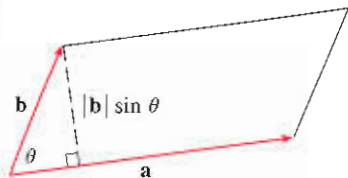The area A of the triangle PQR is half the area of this parallelogram, that is, $\frac{5}{2}\sqrt{82}$.

If we apply Theorems 5 and 6 to the standard basis vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ using $\theta = \pi/2$, we obtain

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i} \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k} \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

Observe that

$$\mathbf{i} \times \mathbf{j} \neq \mathbf{j} \times \mathbf{i}$$

Thus, the cross product is not commutative. Also

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

whereas

$$(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0}$$

So the associative law for multiplication does not usually hold; that is, in general,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

However, some of the usual laws of algebra *do* hold for cross products. The following theorem summarizes the properties of vector products.

---

8 Theorem If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors and c is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

These properties can be proved by writing the vectors in terms of their components and using the definition of a cross product. We give the proof of Property 5 and leave the remaining proofs as exercises.

Proof of Property 5 If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, and $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$, then

$$\begin{aligned}
\boxed{9} \quad \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= a_1(b_2 c_3 - b_3 c_2) + a_2(b_3 c_1 - b_1 c_3) + a_3(b_1 c_2 - b_2 c_1) \\
&= a_1 b_2 c_3 - a_1 b_3 c_2 + a_2 b_3 c_1 - a_2 b_1 c_3 + a_3 b_1 c_2 - a_3 b_2 c_1 \\
&= (a_2 b_3 - a_3 b_2) c_1 + (a_3 b_1 - a_1 b_3) c_2 + (a_1 b_2 - a_2 b_1) c_3 \\
&= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}
\end{aligned}$$

The product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ that occurs in Property 5 is called the **scalar triple product** of the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. Notice from Equation 9 that we can write the scalar triple product as a determinant:

$$\boxed{10} \qquad \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$
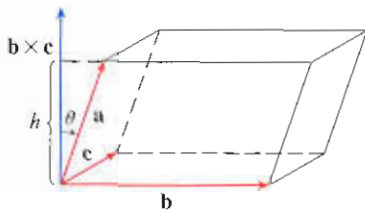
FIGURE 3

The geometric significance of the scalar triple product can be seen by considering the parallelepiped determined by the vectors **a**, **b**, and **c** (Figure 3). The area of the base parallelogram is $A = |\mathbf{b} \times \mathbf{c}|$. If θ is the angle between **a** and $\mathbf{b} \times \mathbf{c}$, then the height h of the parallelepiped is $h = |\mathbf{a}||\cos\theta|$. (We must use $|\cos\theta|$ instead of $\cos\theta$ in case $\theta > \pi/2$.) Therefore, the volume of the parallelepiped is

$$V = Ah = |\mathbf{b} \times \mathbf{c}||\mathbf{a}||\cos\theta| = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

Thus, we have proved the following formula.

---

11 The volume of the parallelepiped determined by the vectors **a**, **b**, and **c** is the magnitude of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

If we use the formula in (11) and discover that the volume of the parallelepiped determined by **a**, **b**, and **c** is 0, then the vectors must lie in the same plane; that is, they are **coplanar**.

EXAMPLE 5 Use the scalar triple product to show that the vectors $\mathbf{a} = \langle 1, 4, -7 \rangle$, $\mathbf{b} = \langle 2, -1, 4 \rangle$, and $\mathbf{c} = \langle 0, -9, 18 \rangle$ are coplanar.

SOLUTION We use Equation 10 to compute their scalar triple product:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 4 \begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} - 7 \begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix}$$

$$= 1(18) - 4(36) - 7(-18) = 0$$

Therefore, by (11) the volume of the parallelepiped determined by **a**, **b**, and **c** is 0. This means that **a**, **b**, and **c** are coplanar.

The idea of a cross product occurs often in physics. In particular, we consider a force **F** acting on a rigid body at a point given by a position vector **r**. (For instance, if we tighten a bolt by applying a force to a wrench as in Figure 4, we produce a turning effect.) The **torque** τ (relative to the origin) is defined to be the cross product of the position and force vectors

$$\tau = \mathbf{r} \times \mathbf{F}$$

FIGURE 4

and measures the tendency of the body to rotate about the origin. The direction of the torque vector indicates the axis of rotation. According to Theorem 6, the magnitude of the torque vector is

$$|\tau| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}|\sin\theta$$

where θ is the angle between the position and force vectors. Observe that the only component of **F** that can cause a rotation is the one perpendicular to **r**, that is, $|\mathbf{F}|\sin\theta$. The magnitude of the torque is equal to the area of the parallelogram determined by **r** and **F**.

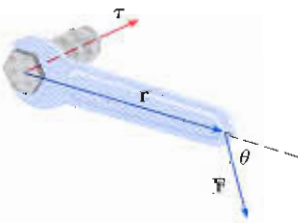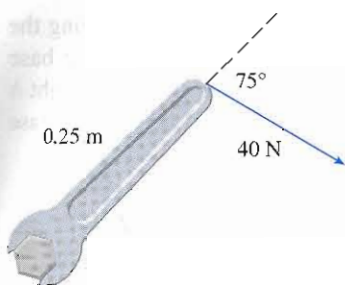**EXAMPLE 6** A bolt is tightened by applying a 40-N force to a 0.25-m wrench as shown in Figure 5. Find the magnitude of the torque about the center of the bolt.

SOLUTION The magnitude of the torque vector is

$$|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}| \sin 75° = (0.25)(40) \sin 75°$$
$$= 10 \sin 75° \approx 9.66 \text{ N·m} = 9.66 \text{ J}$$

If the bolt is right-threaded, then the torque vector itself is

$$\boldsymbol{\tau} = |\boldsymbol{\tau}| \mathbf{n} \approx 9.66 \mathbf{n}$$

where **n** is a unit vector directed down into the page.

FIGURE 5

12.4 Exercises

1–7 ▌ Find the cross product $\mathbf{a} \times \mathbf{b}$ and verify that it is orthogoual to both **a** and **b**.

1. $\mathbf{a} = \langle 1, 2, 0 \rangle$, $\mathbf{b} = \langle 0, 3, 1 \rangle$

2. $\mathbf{a} = \langle 5, 1, 4 \rangle$, $\mathbf{b} = \langle -1, 0, 2 \rangle$

3. $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{b} = \mathbf{j} + 2\mathbf{k}$

4. $\mathbf{a} = \mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

5. $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{b} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$

6. $\mathbf{a} = \mathbf{i} + e^t \mathbf{j} + e^{-t} \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + e^t \mathbf{j} - e^{-t} \mathbf{k}$

7. $\mathbf{a} = \langle t, t^2, t^3 \rangle$, $\mathbf{b} = \langle 1, 2t, 3t^2 \rangle$

8. If $\mathbf{a} = \mathbf{i} - 2\mathbf{k}$ and $\mathbf{b} = \mathbf{j} + \mathbf{k}$, find $\mathbf{a} \times \mathbf{b}$. Sketch **a**, **b**, and $\mathbf{a} \times \mathbf{b}$ as vectors starting at the origin.

9. State whether each expression is meaningfnl. If not, explain why. If so, state whether it is a vector or a scalar.
(a) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ (b) $\mathbf{a} \times (\mathbf{b} \cdot \mathbf{c})$
(c) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ (d) $(\mathbf{a} \cdot \mathbf{b}) \times \mathbf{c}$
(e) $(\mathbf{a} \cdot \mathbf{b}) \times (\mathbf{c} \cdot \mathbf{d})$ (f) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$
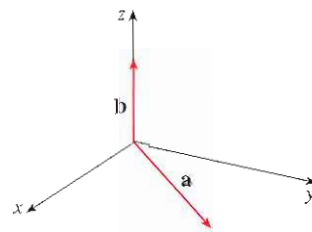
10–11 ▌ Find $|\mathbf{u} \times \mathbf{v}|$ and determine whether $\mathbf{u} \times \mathbf{v}$ is directed into the page or out of the page.

10.

$|\mathbf{u}| = 5$
$60°$
$|\mathbf{v}| = 10$

11.

$|\mathbf{u}| = 6$
$|\mathbf{v}| = 8$
$150°$
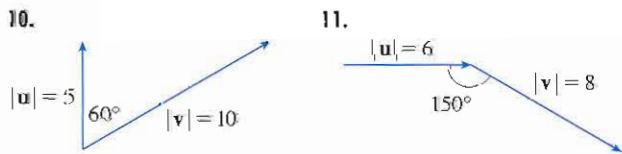
12. The fignre shows a vector **a** in the xy-plane and a vector **b** in the direction of **k**. Their lengths are $|\mathbf{a}| = 3$ and $|\mathbf{b}| = 2$.
(a) Find $|\mathbf{a} \times \mathbf{b}|$.

(b) Use the right-hand rule to decide whether the components of $\mathbf{a} \times \mathbf{b}$ are positive, negative, or 0.

13. If $\mathbf{a} = \langle 1, 2, 1 \rangle$ and $\mathbf{b} = \langle 0, 1, 3 \rangle$, find $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$.

14. If $\mathbf{a} = \langle 3, 1, 2 \rangle$, $\mathbf{b} = \langle -1, 1, 0 \rangle$, and $\mathbf{c} = \langle 0, 0, -4 \rangle$, show that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

15. Find two unit vectors orthogonal to both $\langle 1, -1, 1 \rangle$ and $\langle 0, 4, 4 \rangle$.

16. Find two unit vectors orthogonal to both $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $2\mathbf{i} + \mathbf{k}$.

17. Show that $\mathbf{0} \times \mathbf{a} = \mathbf{0} = \mathbf{a} \times \mathbf{0}$ for any vector **a** in V_3.

18. Show that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$ for all vectors **a** and **b** in V_3.

19. Prove Property 1 of Theorem 8.

20. Prove Property 2 of Theorem 8.

21. Prove Property 3 of Theorem 8.

22. Prove Property 4 of Theorem 8.

23. Find the area of the parallelogram with vertices $A(-2, 1)$, $B(0, 4)$, $C(4, 2)$, and $D(2, -1)$.

24. Find the area of the parallelogram with vertices $K(1, 2, 3)$, $L(1, 3, 6)$, $M(3, 8, 6)$, and $N(3, 7, 3)$.

25–28 ▌ (a) Find a vector orthogonal to the plane throngh the points P, Q, and R, and (b) find the area of triangle PQR.

25. $P(1, 0, 0)$, $Q(0, 2, 0)$, $R(0, 0, 3)$

26. $P(2, 1, 5)$, $Q(-1, 3, 4)$, $R(3. 0, 6)$

27. $P(0, -2, 0)$, $Q(4, 1, -2)$, $R(5, 3, 1)$

28. $P(2, 0, -3)$, $Q(3, 1, 0)$, $R(5, 2, 2)$

29–30 ⅲ Find the volume of the parallelepiped determined by the vectors **a**, **b**, and c.

29. $\mathbf{a} = \langle 6, 3, -1 \rangle$, $\mathbf{b} = \langle 0, 1, 2 \rangle$, $\mathbf{c} = \langle 4, -2, 5 \rangle$

30. $\mathbf{a} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{c} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$

31–32 ⅲ Find the volume of the parallelepiped with adjacent edges *PQ*, *PR*, and *PS*.

31. $P(2, 0, -1)$, $Q(4, 1, 0)$, $R(3, -1, 1)$, $S(2, -2, 2)$

32. $P(0, 1, 2)$, $Q(2, 4, 5)$, $R(-1, 0, 1)$, $S(6, -1, 4)$

33. Use the scalar triple product to verify that the vectors $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}, \mathbf{b} = \mathbf{i} - \mathbf{j}$, and $\mathbf{c} = 7\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ are coplanar.

34. Use the scalar triple product to determine whether the points $P(1, 0, 1)$, $Q(2, 4, 6)$, $R(3, -1, 2)$, and $S(6, 2, 8)$ lie in the same plane.

35. A bicycle pedal is pushed by a foot with a 60-N force as shown. The shaft of the pedal is 18 cm long. Find the magnitude of the torque about *P*.
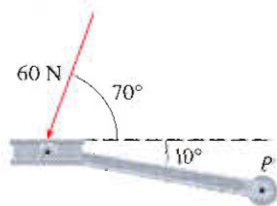
36. Find the magnitude of the torque about *P* if a 36-lb force is applied as shown.
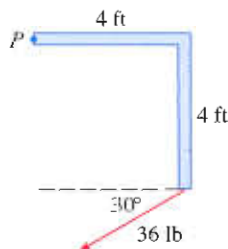
37. A wrench 30 cm long lies along the positive *y*-axis and grips a bolt at the origin. A force is applied in the direction $\langle 0, 3, -4 \rangle$ at the end of the wrench. Find the magnitude of the force needed to supply 100 J of torque to the bolt.

38. Let $\mathbf{v} = 5\mathbf{j}$ and let **u** be a vector with length 3 that starts at the origin and rotates in the *xy*-plane. Find the maximum and

minimum values of the length of the vector $\mathbf{u} \times \mathbf{v}$. In what direction does $\mathbf{u} \times \mathbf{v}$ point?

39. (a) Let *P* be a point not on the line *L* that passes through the points *Q* and *R*. Show that the distance *d* from the point *P* to the line *L* is

$$d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|}$$

where $\mathbf{a} = \overrightarrow{QR}$ and $\mathbf{b} = \overrightarrow{QP}$.

(b) Use the formula in part (a) to find the distance from the point $P(1, 1, 1)$ to the line through $Q(0, 6, 8)$ and $R(-1, 4, 7)$.

40. (a) Let *P* be a point not on the plane that passes through the points *Q*, *R*, and *S*. Show that the distance *d* from *P* to the plane is

$$d = \frac{|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|}{|\mathbf{a} \times \mathbf{b}|}$$

where $\mathbf{a} = \overrightarrow{QR}$, $\mathbf{b} = \overrightarrow{QS}$, and $\mathbf{c} = \overrightarrow{QP}$.

(b) Use the formula in part (a) to find the distance from the point $P(2, 1, 4)$ to the plane through the points $Q(1, 0, 0)$, $R(0, 2, 0)$, and $S(0, 0, 3)$.

41. Prove that $(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2(\mathbf{a} \times \mathbf{b})$.

42. Prove part 6 of Theorem 8, that is,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

43. Use Exercise 42 to prove that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0$$

44. Prove that

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \mathbf{a} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{c} \\ \mathbf{a} \cdot \mathbf{d} & \mathbf{b} \cdot \mathbf{d} \end{vmatrix}$$

45. Suppose that $\mathbf{a} \neq 0$.

(a) If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$?

(b) If $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$?

(c) If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ and $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$?

46. If $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are noncoplanar vectors, let

$$\mathbf{k}_1 = \frac{\mathbf{v}_2 \times \mathbf{v}_3}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)} \qquad \mathbf{k}_2 = \frac{\mathbf{v}_3 \times \mathbf{v}_1}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}$$

$$\mathbf{k}_3 = \frac{\mathbf{v}_1 \times \mathbf{v}_2}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}$$

(These vectors occur in the study of crystallography. Vectors of the form $n_1\mathbf{v}_1 + n_2\mathbf{v}_2 + n_3\mathbf{v}_3$, where each n_i is an integer, form a *lattice* for a crystal. Vectors written similarly in terms of $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$ form the *reciprocal lattice*.)

(a) Show that $\mathbf{k}_i$ is perpendicular to $\mathbf{v}_j$ if $i \neq j$.

(b) Show that $\mathbf{k}_i \cdot \mathbf{v}_i = 1$ for $i = 1, 2, 3$.

(c) Show that $\mathbf{k}_1 \cdot (\mathbf{k}_2 \times \mathbf{k}_3) = \dfrac{1}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}$.