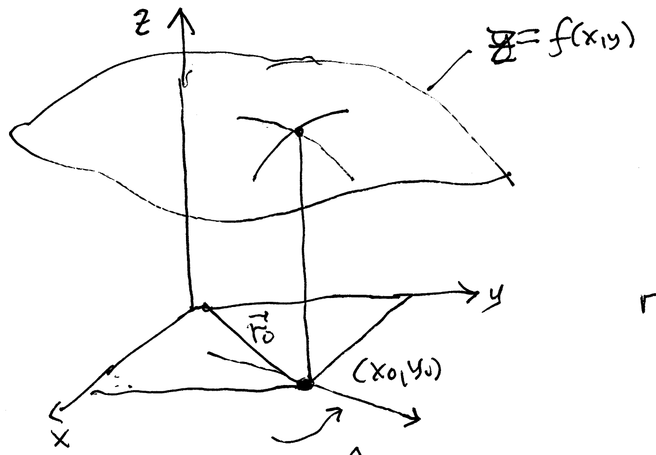
2nd derivative test in 2D



$$\vec{r} = \vec{r_0} + s\hat{u}$$

$$\frac{d\vec{r}}{ds} = \hat{u} = \langle u_1, u_2 \rangle$$

2nd directional derivative:

$$f(\vec{r}) = f(\vec{r_0} + s\hat{u})$$

$$\frac{df(\vec{r}(s))}{ds} = \frac{\partial f}{\partial x}(\vec{r}(s))\underbrace{\frac{dx}{ds}}_{u_1} + \frac{\partial f}{\partial y}(\vec{r}(s))\underbrace{\frac{dy}{ds}}_{u_2}$$

$$= (\hat{u}\cdot\vec{\nabla} f)(\vec{r}(s))$$

repeat:

$$\frac{d^2}{ds^2} f(\vec{r}(s)) = (\hat{u}\cdot\vec{\nabla})(\hat{u}\cdot\vec{\nabla} f)(\vec{r}(s))$$

$$\left.\frac{d^2 f}{ds^2}(\vec{r}(s))\right|_{s=0} = (\hat{u}\cdot\vec{\nabla})(\hat{u}\cdot\vec{\nabla} f)(\vec{r_0})$$

$$= \left(u_1\frac{\partial}{\partial x} + u_2\frac{\partial}{\partial y}\right)\left(u_1\frac{\partial f}{\partial x} + u_2\frac{\partial f}{\partial y}\right)\Big|_{\vec{r_0}}$$

$$= \left(u_1^2\frac{\partial^2 f}{\partial x^2} + 2u_1u_2\frac{\partial^2 f}{\partial x\partial y} + u_2^2\frac{\partial^2 f}{\partial y^2}\right)\Big|_{\vec{r_0}}$$

$$= f_{xx}(\vec{r_0})u_1^2 + 2f_{xy}(\vec{r_0})u_1u_2 + f_{yy}(\vec{r_0})u_2^2 \overset{\text{set}}{=} 0$$

at a critical pt where tangent plane is horizontal: $\vec{\nabla} f(\vec{r_0}) = 0$

or $\frac{\partial f}{\partial x}(x_0,y_0) = 0 = \frac{\partial f}{\partial y}(x_0,y_0)$

want 2nd derivative to have same sign in all directions to be a local extremum

$f_{xx}(\vec{r_0})$ and $f_{yy}(\vec{r_0})$ must both be of the same sign to have consistent extrema.

If $f_{xx}(\vec{r_0})f_{yy}(\vec{r_0}) - f_{xy}(\vec{r_0})^2 > 0$ confirms this guess

if $f_{xx}(\vec{r_0})f_{yy}(\vec{r_0}) - f_{xy}(\vec{r_0})^2 \begin{cases} = 0 & \text{test fails} \\ < 0 & \text{saddle} \end{cases}$

$$\underset{\text{cant change sign}}{u_2^2}\underbrace{\left[\overset{A}{f_{xx}(\vec{r_0})}\left(\frac{u_1}{u_2}\right)^2 + \overset{B}{2f_{xy}(\vec{r_0})}\left(\frac{u_1}{u_2}\right) + \overset{C}{f_{yy}(\vec{r_0})}\right]}_{\text{quadratic equation if } =0:} = 0$$

$$A\left(\frac{u_1}{u_2}\right)^2 + B\left(\frac{u_1}{u_2}\right) + C = 0$$

$$\frac{u_1}{u_2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$B^2 - 4AC$ "discriminant"

want discriminant < 0 so complex roots, then cannot change sign (must pass through zero to change sign)

or $-\frac{\text{discriminant}}{4} > 0$

$$\underbrace{AC - \frac{B^2}{4}}_{} = \underbrace{f_{xx}(\vec{r_0})f_{yy}(\vec{r_0}) - f_{xy}(\vec{r_0})^2 > 0}_{\text{condition for consistent sign of 2nd derivative in all directions.}}$$

note: $f_{xx}(\vec{r_0})f_{yy}(\vec{r_0}) > 0$ if both are of same sign

If subtracting the mixed derivative term does not lead to a negative number (still positive), then guess based on consistent sign of $f_{xx}(\vec{r_0})$ and $f_{yy}(\vec{r_0})$ is valid in all directions