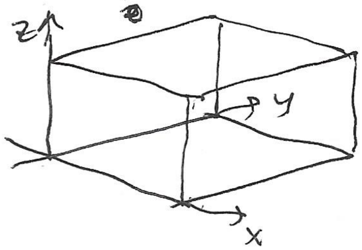
Stewart Calculus 8e, 14.7.53 (Max-Min word problems)

A cardboard box without a lid is to have a volume V cm³.
Find the dimensions that minimizes the amount of cardboard used.
Let x, y, z be the dimensions of the box. [Report the ~~A~~ minimal area as well.]

Draw picture:



$V = xyz = \text{constant}$ (CONSTRAINT)

$A = \underbrace{xy}_{\text{bottom}} + \underbrace{2z(x+y)}_{\text{2 pairs of sides}}$ (FUNCTION TO BE EXTREMIZED)

solve for z: $z = \frac{V}{xy} > 0$

eliminate from A: $A = xy + 2\left(\frac{V}{xy}\right)(x+y)$

2d math problem: $\boxed{A(x,y) = xy + \frac{2V(x+y)}{xy}, \quad x>0, y>0,}$

$A \to \infty$ as $x \to 0$ or $y \to 0$

$A > 0$

$$0 = A_x = y + 2V\,\frac{xy(1) - (x+y)y}{x^2y^2} = y - \frac{2Vy^2}{x^2y^2} = y - \frac{2V}{x^2} \rightarrow y = \frac{2V}{x^2}$$

$$0 = A_y = x + 2V\left(\frac{xy(1) - (x+y)x}{x^2y^2}\right) = x - \frac{2Vx^2}{x^2y^2} = x - \frac{2V}{y^2}$$

$$0 = x - 2V\left(\frac{x^2}{2V}\right)^2 = x - \frac{x^4}{(2V)} = x\left(1 - \frac{x^3}{(2V)}\right) \rightarrow x^3 = 2V$$

$$x = (2V)^{1/3}$$

$$y = \frac{2V}{((2V)^{1/3})^2} = \frac{2V}{(2V)^{2/3}} = (2V)^{1/3}$$

$$z = \frac{V}{(2V)^{1/3}(2V)^{1/3}} = \frac{(2V)}{2(2V)^{2/3}} = \frac{1}{2}(2V)^{1/3}$$

$$[x,y,z] = \underbrace{(2V)^{1/3}}_{\text{sets scale}}\left[1, 1, \tfrac{1}{2}\right]$$

$$A\left((2V)^{1/3}, (2V)^{1/3}\right) = (2V)^{1/3}(2V)^{1/3} + 2V\,\frac{\left((2V)^{1/3} + (2V)^{1/3}\right)}{(2V)^{1/3}(2V)^{1/3}}$$

$$= (2V)^{2/3} + (2V)^{1/3} \cdot 2(2V)^{1/3} = \boxed{3(2V)^{2/3} \text{ cm}^2}$$

If $V = 4000$ $(2V)^{1/3} = (8000)^{1/3} = \left((20)^3\right)^{1/3} = 20$ so $x = 20, y = 20, z = 10$

$V = 20 \cdot 20 \cdot 10 = 4000$ ✓

$A = 3 \cdot 20^2 = \boxed{1200 \text{ cm}^2}$

The box minimizing the amount of cardboard has equal length and width 20 cm and height 10 cm.