multivariable "antidifferentiation" (potential for conservative vector field)

one independent variable

$$\underbrace{F(x)}_{\text{given}} = \frac{d\underbrace{f(x)}_{\text{to find}}}{dx} \rightarrow f(x) - f(0) = \int_0^{x} \underbrace{F(u)}_{\frac{df(u)}{du}}\,du$$

← variable upper limit

is an integral formula for an antiderivative f modulo the arbitrary constant value $f(0)=C$

multivariable analog (2D)

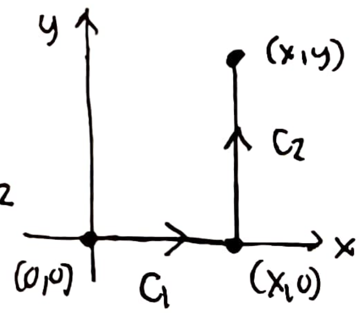
$$\underbrace{\vec{F}(x,y)}_{\text{given}} = \vec{\nabla}\underbrace{f(x,y)}_{\text{to find}} \longrightarrow f(x,y) - f(0,0) = \int_C \vec{F}(x,y)\cdot d\vec{r} \quad \left\{ \begin{array}{l} C: (0,0) \rightarrow \underbrace{(x,y)}_{\text{variable endpoint}} \\ \text{is any curve} \end{array} \right.$$

to find — now called potential function

if it exists then it doesn't matter what curve we use to get from our reference point [(0,0) for simplicity] to a general point (x,y)

so we can use two successive partial integrations moving along one coordinate line and then another:

$C = C_1 + C_2$



C_1: $\vec{r}(t) = \langle t, 0\rangle \quad \vec{r}\,'(t) = \langle 1, 0\rangle \qquad t = 0..x$

$\vec{F}(\vec{r}(t)) = \langle 3, x^2\rangle \quad \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = 3$

$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_0^x 3\,dt = 3t\Big|_0^x = \boxed{3x}$

C_2: $\vec{r} = \langle x, t\rangle \quad \vec{r}\,'(t) = \langle 0, 1\rangle \qquad t = 0..y$

$\vec{F}(\vec{r}(t)) = \langle 3 + 2xt, x^2 - 3t^2\rangle \quad \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = x^2 - 3t^2$

$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_0^y x^2 - 3t^2\,dt = x^2t - t^3\Big|_0^y = \boxed{x^2y - y^3}$

$\int_C \vec{F}\cdot d\vec{r} = 3x + x^2y - y^2$ so

$f(x,y) = \underbrace{f(0,0)}_{C} + 3x + x^2y - y^2$

$$\vec{F}(x,y) = \langle 3 + 2xy, x^2 - 3y^2\rangle$$

$\frac{\partial}{\partial y} \to 2x \qquad \frac{\partial}{\partial x} \to 2x$

$\therefore$ conservative

PDE approach

$$\left.\begin{array}{l} \frac{\partial f}{\partial x} = F_1(x,y) \\ \frac{\partial f}{\partial y} = F_2(x,y) \end{array}\right\}$$

is a system of Partial Differential Equations for an unknown function $f(x,y)$ but if it exists, its mixed 2nd partials must agree:

$$\begin{array}{l} \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}F_1 \\ \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(F_2) \end{array}$$

must be equal

$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$

called an integrability condition for the PDE system (condition for being able to "integrate" the system, i.e., solve it by integration techniques).

An equivalent partial integration technique can be used to solve one of these two equations & then backsub into the second to require it to be satisfied:

$$\int \left[\frac{\partial f}{\partial x} = 3 + 2xy\right] dx \rightarrow f = \int 3 + 2xy\,dx = 3x + x^2y + C(y)$$

← "constant" may depend on y held fixed

$$\frac{\partial f}{\partial y} = x^2 - 3y^2 \rightarrow \underbrace{\frac{\partial}{\partial y}(3x + x^2y + C(y))}_{x^2 + \frac{dC(y)}{dy}} = x^2 - 3y^2 \quad \Big\} \quad \frac{dC(y)}{dy} = -3y^2 \rightarrow C(y) = -y^3 + k$$

so $\boxed{f = 3x + x^2y - y^3 + k.}$ same result.