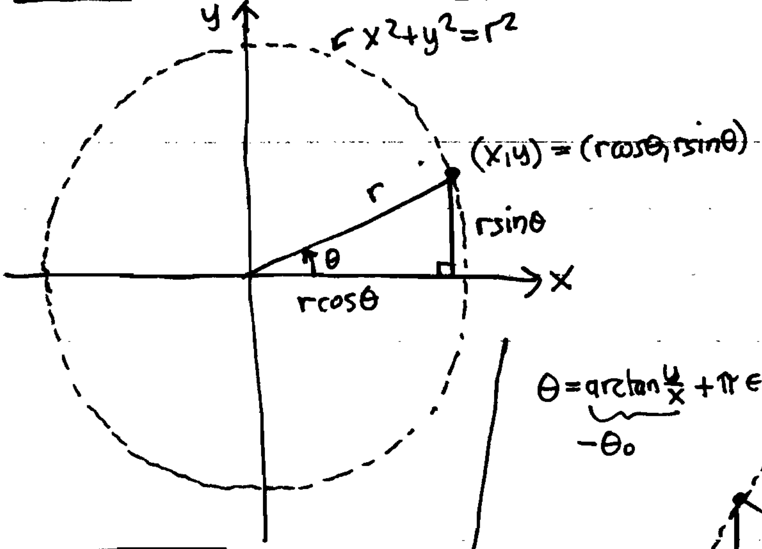
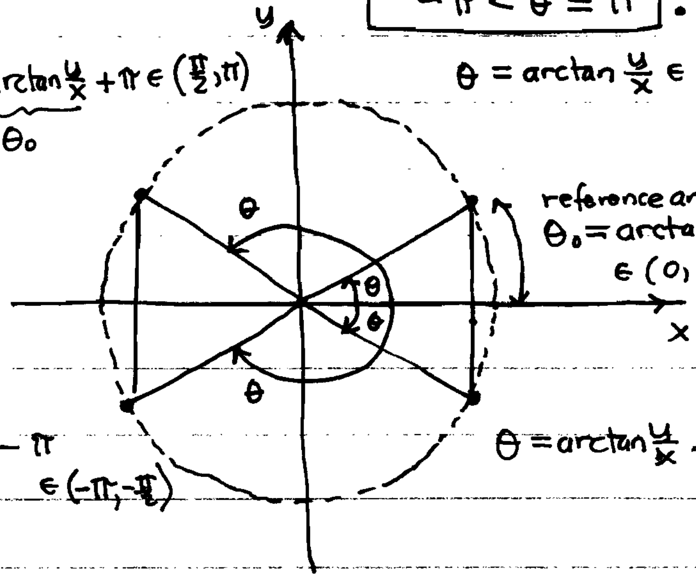
polar coordinates in the plane



usually $r \geq 0$ but for some curves $r = f(\theta)$ it is convenient to allow $r<0$.

$\theta = 0..2\pi$ or $\theta = -\pi..\pi$ are two natural ranges for the interval of theta values.

$\boxed{-\pi < \theta \leq \pi}$:



$\theta = \underbrace{\arctan\frac{y}{x}}_{-\theta_0} + \pi \in (\frac{\pi}{2}, \pi)$

$\theta = \arctan\frac{y}{x} \in (0, \frac{\pi}{2})$

reference angle $\theta_0 = \arctan\left|\frac{y}{x}\right| \in (0, \frac{\pi}{2})$

$\theta = \underbrace{\arctan\frac{y}{x}}_{\theta_0} - \pi \in (-\pi, -\frac{\pi}{2})$

$\theta = \arctan\frac{y}{x} \in (-\frac{\pi}{2}, 0)$

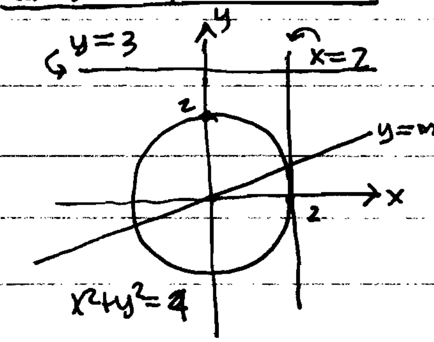
$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

$+^2$: $x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$
$= r^2(\cos^2\theta+\sin^2\theta) = r^2$

$\therefore\ r = \sqrt{x^2+y^2} \geq 0$ (usually)

$\div$: $\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta \xrightarrow[\text{not just arctan}]{} $ solve not just arctan

cartesian to polar conversion



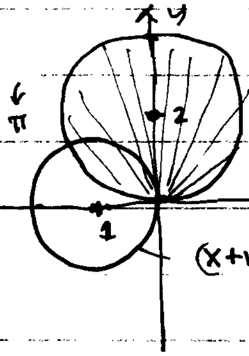
lines

vertical:
$x=2 \rightarrow r\cos\theta = 2 \rightarrow r = 2/\cos\theta = 2\sec\theta$

horizontal:
$y=3 \rightarrow r\sin\theta = 3 \rightarrow r = 3/\sin\theta = 3\csc\theta$

slanted thru origin:
$y=mx \leftrightarrow m = \frac{y}{x} = \tan\theta \rightarrow \theta = \arctan m$

$x^2+y^2=4 \rightarrow r^2=4 \rightarrow r=2$ ($\theta = 0..2\pi$ is one loop)



$x^2+(y-2)^2 = 2^2 \rightarrow \underbrace{x^2+y^2}_{r^2} - 4y + \cancel{4} = \cancel{4} \rightarrow r^2 = 4r\sin\theta \rightarrow r = 4\sin\theta$

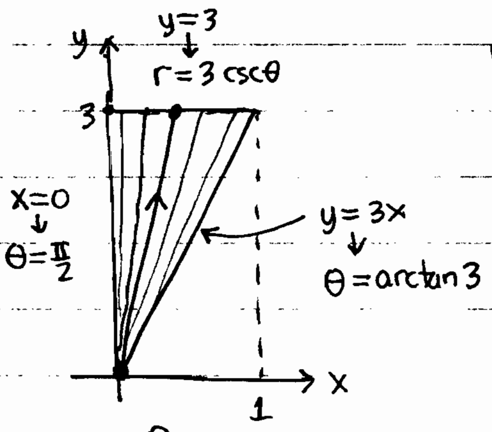
$(x+1)^2+y^2=1^2 \rightarrow \underbrace{x^2+y^2}_{r^2} + 2x + \cancel{1} = \cancel{1} \rightarrow r^2 = -2r\cos\theta \rightarrow r = -2\cos\theta$

in both cases $\theta = 0..\pi$ and $\theta = \frac{\pi}{2}..\frac{3\pi}{2}$ respectively makes one loop of $r \geq 0$, while the second half of $\theta = 0..2\pi$ adds a second loop of $r \leq 0$, repeating the same path.

for integration purposes we will need to describe bounding curves of regions with $r \geq 0$ and no repeating.

polar coordinate integration examples

same function $f(x,y) = x^2+y^2 = r^2$ on two regions R_1 and R_2



R_1:

$r = 0 .. 3\csc\theta$

for $\theta = \arctan 3 .. \frac{\pi}{2}$

$$\iint_{R_1} f(x,y)\,dA = \int_{\arctan 3}^{\pi/2}\int_0^{3\csc\theta} \overbrace{\underbrace{(r^2)}_{f(x,y)}\ \underbrace{r\,dr\,d\theta}_{dA}}^{r^3}$$

$$= \int_{\arctan 3}^{\pi/2} \frac{r^4}{4}\Big|_{r=0}^{r=3\csc\theta} d\theta = \int_{\arctan 3}^{\pi/2} \frac{3^4}{4}\csc^4\theta\, d\theta$$

TECHNOLOGY:

$$\int \csc^4\theta\, d\theta = \underbrace{-\frac{1}{3}\frac{\cos\theta}{\sin 3\theta} - \frac{2}{3}\frac{\cos\theta}{\sin\theta}}_{\text{more useful}} + C = \underbrace{-\frac{1}{3}\cot\theta\csc^2\theta - \frac{2}{3}\cot\theta + C}_{\text{more elegant}}$$

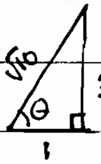
$$= \frac{3^4}{4}\left(-\frac{1}{3}\frac{\overset{0}{\cancel{\cos\pi/2}}}{\sin^3\pi/2} - \frac{2}{3}\frac{\overset{0}{\cancel{\cos\pi/2}}}{\sin\pi/2}\right) - \frac{3^4}{4}\left(\frac{1}{3}\underbrace{\frac{\cos(\arctan 3)}{\sin^3(\arctan 3)}} - \frac{2}{3}\underbrace{\frac{\cos(\arctan 3)}{\sin(\arctan 3)}}\right)\Bigg]$$

$\cos\pi/2 = 0$

$\sin\pi/2 = 1$

$$\frac{1/\sqrt{10}}{(3/\sqrt{10})^3} \qquad \frac{1/\sqrt{10}}{3/\sqrt{10}} = \frac{1}{3}$$

$$\frac{10}{3^3}$$



$\theta = \arctan 3$

$\cot\theta = \frac{1}{3}$

$\csc\theta = \frac{\sqrt{10}}{3}$

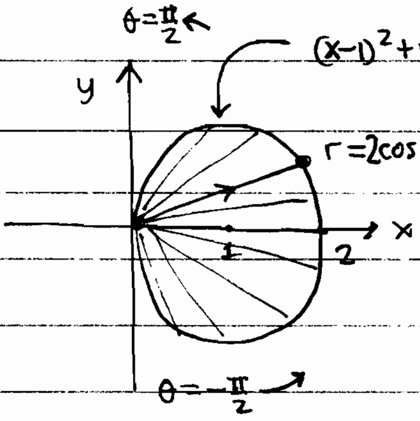
or

$\cos\theta = 1/\sqrt{10}$

$\sin\theta = 3/\sqrt{10}$

how one calculates with inverse trig functions

$$= \frac{3^4}{4}\left[\frac{10}{3^4} + \frac{2}{9}\right] = \frac{1}{2}(5+9) = \boxed{7}$$



R_2: $r = 0 .. 2\cos\theta$

for $\theta = -\frac{\pi}{2} .. \frac{\pi}{2}$

→ ~~$(x-1)^2+y^2=1$ circle of radius 1, center at (1,0)~~

$x^2-2x+1+y^2=1$

$x^2+y^2-2x=0$

$r^2 - 2r\cos\theta = 0$

$r - 2\cos\theta = 0$

$r = 2\cos\theta$

$$\iint_{R_2} f(x,y)\,dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} \overbrace{\underbrace{(r^2)}_{f(x,y)}\underbrace{(r\,dr\,d\theta)}_{dA}}^{r^3}$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r^4}{4}\Big|_{r=0}^{r=2\cos\theta} d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2^4}{4}\cos^4\theta\, d\theta$$ technology

$$= \frac{2^4}{4}\left[\frac{1}{4}\cos^3\theta\sin\theta + \frac{3}{8}\cos\theta\sin\theta + \frac{3}{8}\theta\right]_{-\pi/2}^{\pi/2}$$

$$= 4\left[0 + 0 + \frac{3}{8}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)\right] = \boxed{\frac{3\pi}{2}}$$

[since $\cos(\pm\frac{\pi}{2}) = 0$]