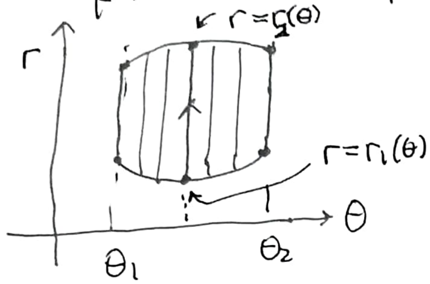
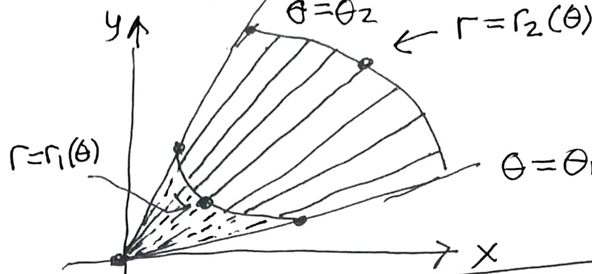
Polar Coordinate Integration (1)

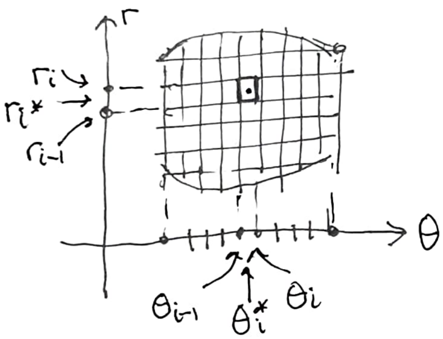
polar coordinate plane

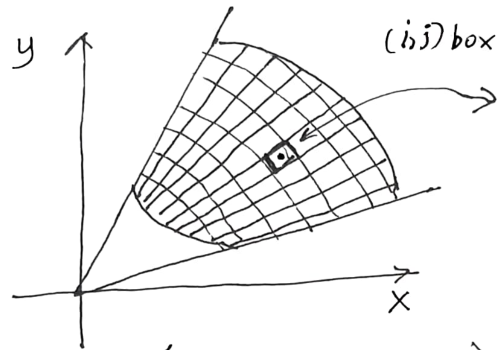


physical x-y plane

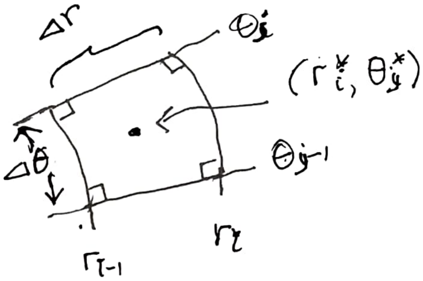


$\boxed{r = r_1(\theta) \,..\, r_2(\theta) \text{ while } \theta = \theta_1 \,..\, \theta_2}$





region R of plane



$\Delta A_{i,j} \approx \underbrace{r_i^* \Delta r \Delta\theta}$

exact if $r_i^* = r_{midpoint} = \frac{1}{2}(r_{i-1}+r_i)$

$\approx \underbrace{(\Delta r)}_{\text{arclength}} \underbrace{(r\Delta\theta)}_{\text{arclength}}$

orthogonal sides

$$\iint_R f\, dA = \lim_{n\to\infty} \lim_{m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} F(r_i^*, \theta_j^*) \underbrace{\Delta A_{ij}}_{r_i^* \Delta r \Delta\theta}$$

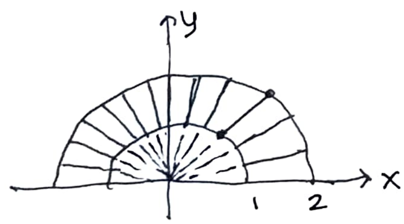
$f(x,y) = f(r\cos\theta, r\sin\theta)$
$\equiv F(r,\theta)$

iterated integral integrand : $F(r,\theta)\, r$

$$= \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} F(r,\theta)\, r\, dr\, d\theta$$

geometric corrective factor : $dA = r\, dr\, d\theta$
$= dr \cdot (r\, d\theta)$

Example (Annulus segment)



$r = 1\,..\,2$ while $\theta = 0\,..\,\pi$

$$\iint \underbrace{3x+4y^2}\, \overbrace{dA}^{r\,dr\,d\theta} = \int_0^{\pi}\int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\, r\,dr\,d\theta$$

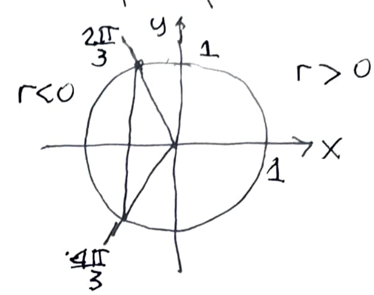
$3(r\cos\theta) + 4(r\sin\theta)^2$
$= 3r\cos\theta + 4r^2\sin^2\theta$

$= \dots = \frac{15\pi}{2} \approx 23.562$

(see Maple worksheet example 4)

Polar Coordinate Integration (3)

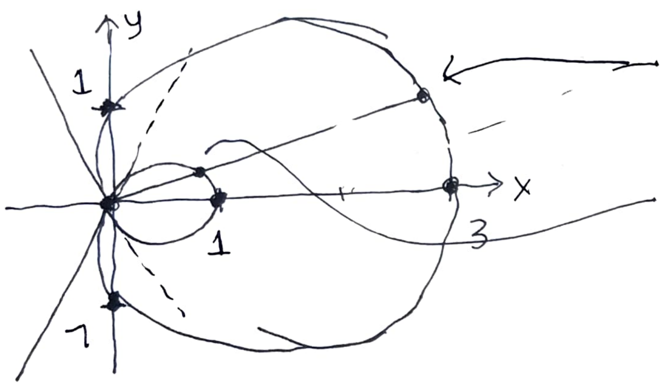
Example. Find area between the inner and outer loops of the cardioid $r = 1 + 2\cos\theta$.



$= 0 \rightarrow \cos\theta = -\frac{1}{2}$

$\theta = -\frac{2\pi}{3} \ldots \frac{2\pi}{3} \quad r > 0$

$\theta = \frac{2\pi}{3} \ldots \frac{4\pi}{3} \quad r < 0$



outer loop

$r = 0 \ldots \underbrace{1 + 2\cos\theta}_{>0}$ while $\theta = -\frac{2\pi}{3} \ldots \frac{2\pi}{3}$

inner loop

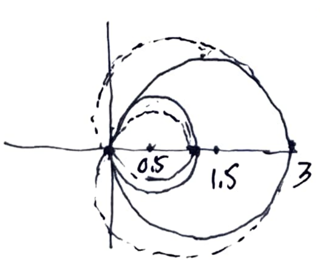
$r = \underbrace{1 + 2\cos\theta}_{<0} \ldots 0$ while $\theta = \frac{2\pi}{3} \ldots \frac{4\pi}{3}$

$$A_{outer} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{1+2\cos\theta} 1 \cdot r\,dr\,d\theta = 2\pi + \frac{3\sqrt{3}}{2}$$

$$A_{inner} = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \int_{1+2\cos\theta}^{0} 1 \cdot \underbrace{(-r)}_{>0!}\,dr\,d\theta = \pi - \frac{3\sqrt{3}}{2}$$

↑ ordered, increasing r

$$A_{between} = A_{outer} - A_{inner} = \pi + 3\sqrt{3} \approx 8.338$$



Guesstimate. Circle Center (1.5, 0) radius 1.5 (too small)

center (0.5, 0) radius 0.5 (a bit bigger)

$\text{Area} \sim \pi(1.5^2 - 0.5^2) \approx 6.28$

in the right ball park!

Polar Coordinate Integration (2)

Example 4. Find volume between $z = \underbrace{x^2+y^2}_{=r^2}$ inside the cylinder $x^2+y^2=2x$

integrand!

solution.

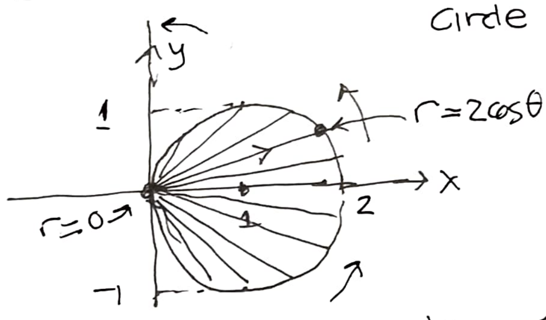
$$\underbrace{x^2+y^2}_{r^2} \underbrace{-2x}_{-2(r\cos\theta)} = 0 \rightarrow (x-1)^2 - 1 + y^2 = 0 \rightarrow (x-1)^2+y^2 = 1$$

Circle center $(1,0)$, radius 1

$r^2 - 2r\cos\theta = 0$

$r(r-2\cos\theta) = 0$

$r = 2\cos\theta$



limits

$r = 0 .. 2\cos\theta$

while $\theta = -\frac{\pi}{2} .. \frac{\pi}{2}$

must be continuous interval.

alternative: $\frac{3\pi}{2} .. 2\pi$

but "too big", use simpler angular range

$$\iint_R x^2+y^2\,dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2 \cdot r\,dr\,d\theta$$

dont forget correction factor!

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r^3}{3}\Big|_{r=0}^{r=2\cos\theta} d\theta = \int_{-\frac{\pi}{2}}^{\pi/2} \frac{2^3}{3}\cos^3\theta\,d\theta = \ldots$$

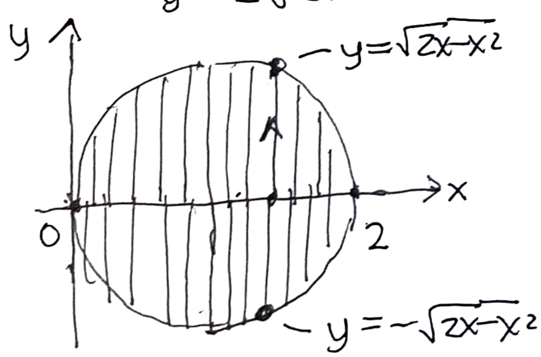
easy with Maple

cartesian iteration.

$x^2+y^2 = 2x$

$y^2 = 2x - x^2$

$y = \pm\sqrt{2x-x^2}$



$$\iint_R x^2+y^2\,dA = \int_0^2 \int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} (x^2+y^2)\,dy\,dx$$

hard integral, need technology to evaluate

(see Maple worksheet)