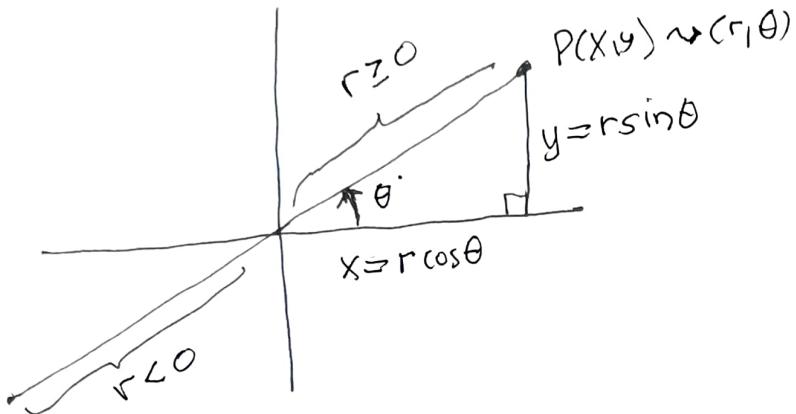


## Polar Coordinates

"grid" (0)



$$r = \sqrt{x^2 + y^2} \geq 0$$

$r < 0$  only for polar curves

$$\tan \theta = y/x$$

If need a unique angle best, choose  
 $-\pi < \theta \leq \pi$

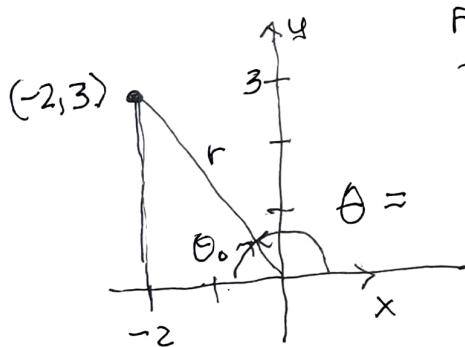
$$\theta \approx \arctan(y/x) = \arctan\left(\frac{y}{x}\right) + \text{const.}$$

Maple       $\in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\pm \pi$  in 2nd/3rd quadrants!

You don't need formulas to convert  
Cartesian to polar coords!

Make a diagram. Use Pythagorean theorem  
and adjust reference angle to quadrant.



First:  $r = \sqrt{(-2)^2 + 3^2} = \sqrt{4+9} = \sqrt{13}$

Then:  $\tan \theta_0 = \frac{3}{2}$ ,  $\theta_0 = \arctan \frac{3}{2}$  (reference angle always acute  $\geq 0$ )  
 $\hookrightarrow \theta = \pi - \theta_0 = \pi - \arctan \frac{3}{2}$

reverse direction:

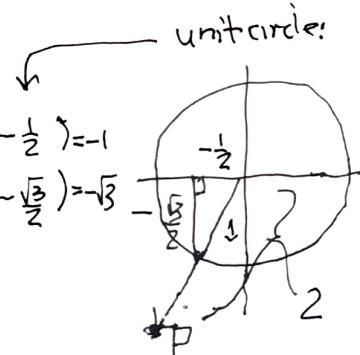
$$(r, \theta) = (2, -\frac{2\pi}{3})$$

$|\theta| > \pi$ ,  $\theta_0 = \frac{2\pi}{3} \sim 60^\circ$   
 in third quadrant

easy to remember from above diagram!

$$x = r \cos \theta = 2 \cos\left(-\frac{2\pi}{3}\right) = 2\left(-\frac{1}{2}\right) = -1$$

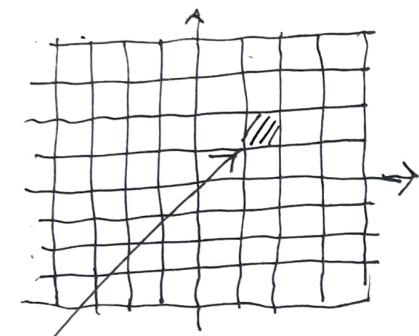
$$y = r \sin \theta = 2 \sin\left(-\frac{2\pi}{3}\right) = 2\left(-\frac{\sqrt{3}}{2}\right) = -\sqrt{3}$$



See Maple worksheet  
arctanyx.mw

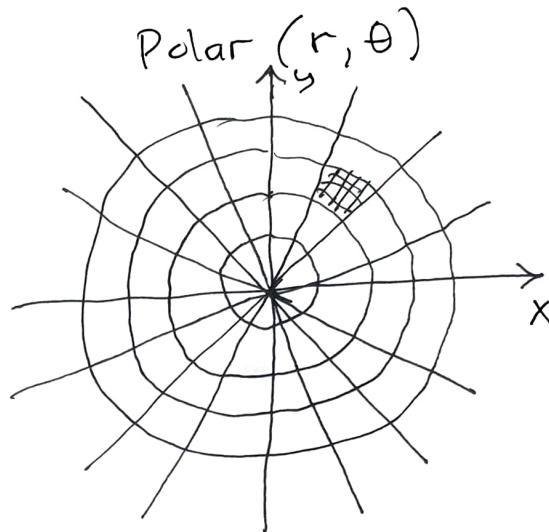
Polar Coordinates : "grid" (1)  
 equally spaced divisions of both coordinates

Cartesian (x,y)



"rectangular grid"  
 grid box is a rectangle

$$\Delta A = \Delta x \Delta y$$



$r$ : Concentric circles  
 $\theta$ : Half rays from origin

grid box is a "sector difference"

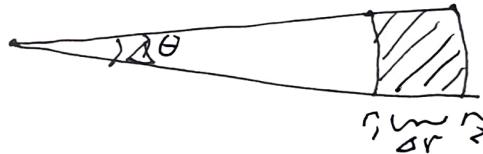
sector area derivation



$$\frac{A_{\text{sector}}}{A_{\text{circle}}} = \frac{\theta}{2\pi}$$

fraction of total pie!

$$\text{solve } A_{\text{sector}} = \frac{1}{2} r^2 \theta$$



$$\Delta A = \frac{1}{2} (r_2^2 - r_1^2) \Delta \theta$$

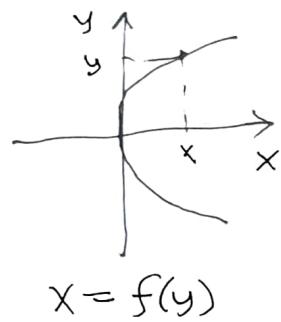
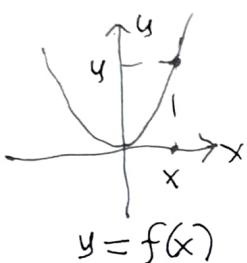
$$= \frac{1}{2} (r_2 + r_1)(r_2 - r_1) \Delta \theta$$

$r_{\text{avg}}$  or  $\Delta r$

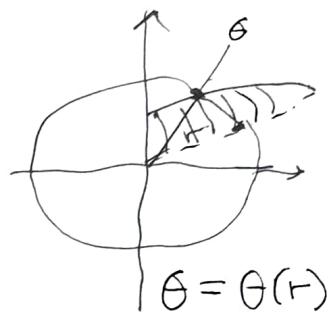
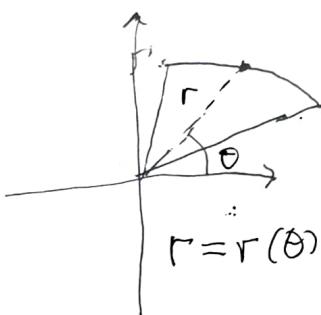
If we want to integrate with respect to the area of a region of the plane in polar coordinates, we need to use this formula for  $\Delta A$  for the Riemann sum over grid boxes. (LATER)

## Polar Coordinates : "grid" (2)

graphs of functional relationships



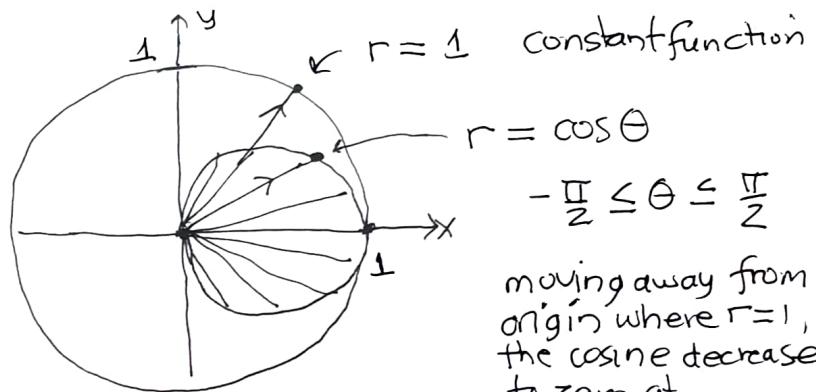
both are useful in integration problems



very useful  
allow  $-\infty \leq \theta \leq \infty$   
to wrap around origin multiple times

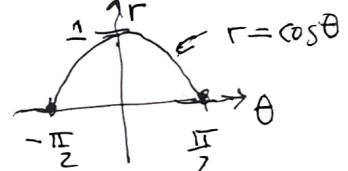
not useful!

### Examples



moving away from origin where  $r=1$ , the cosine decreases to zero at

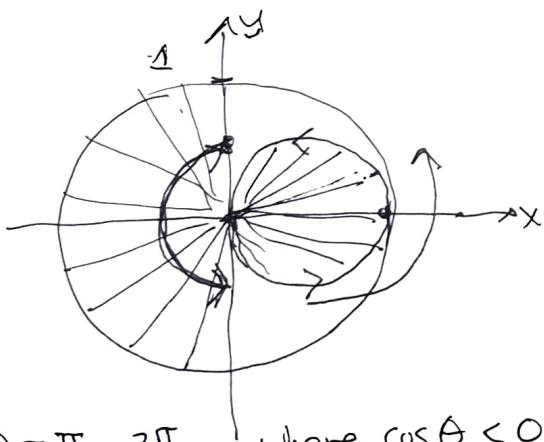
$$\theta = \pm \frac{\pi}{2}$$



$(r, \theta)$  "coordinate" plane

corresponds to closed loop in  $xy$  plane

If extend range of theta, we trace out the same loop multiple times, including where  $\cos \theta < 0$  ( $r < 0$ !)

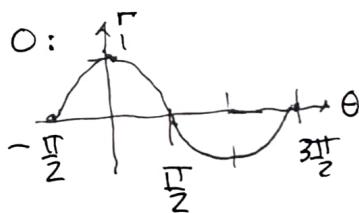


$$\theta = \frac{\pi}{2} \dots \frac{3\pi}{2}$$

theta increases in left half  $xy$  plane

but  $r < 0$  and this

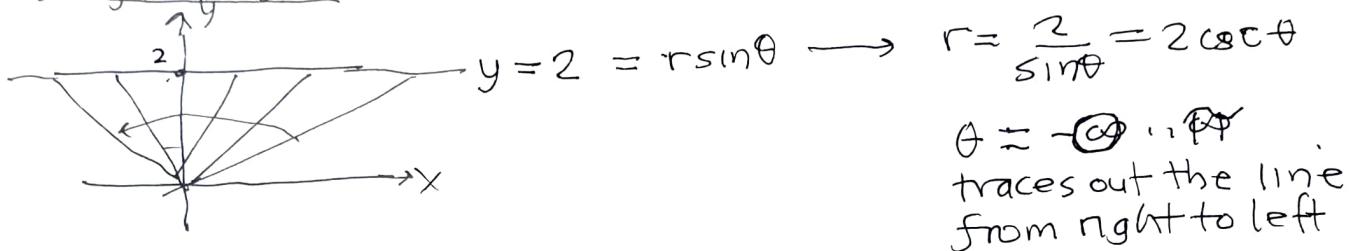
~~traces out the~~ traces out the same loop in the counterclockwise direction in the right half of the  $xy$  plane



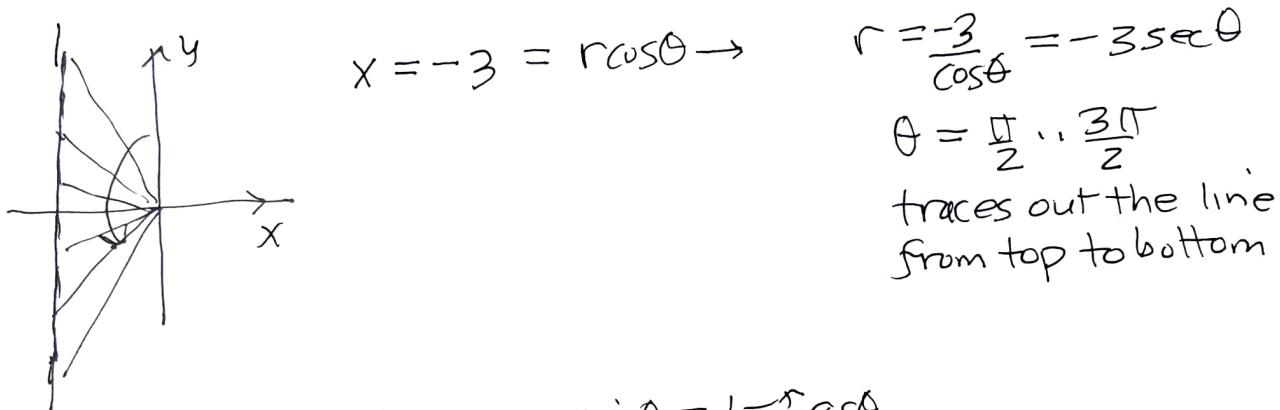
in the right half of the  $xy$  plane

## Polar Coordinates: "grid" (3)

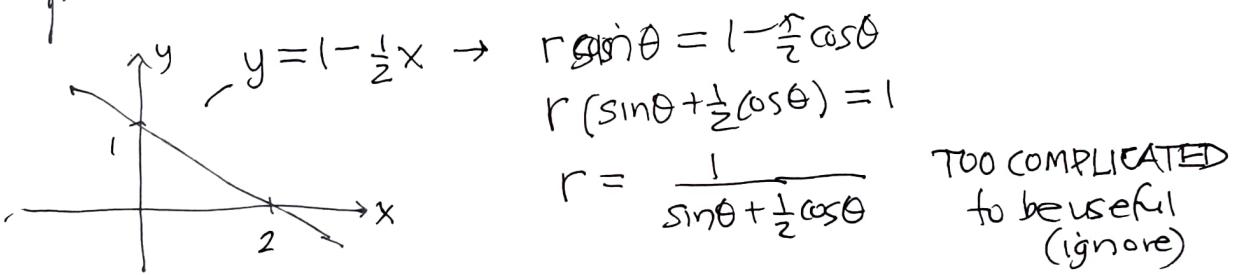
Simple curves in the plane re-expressed in polar coords, as  $r = r(\theta)$ .  
straight lines first?



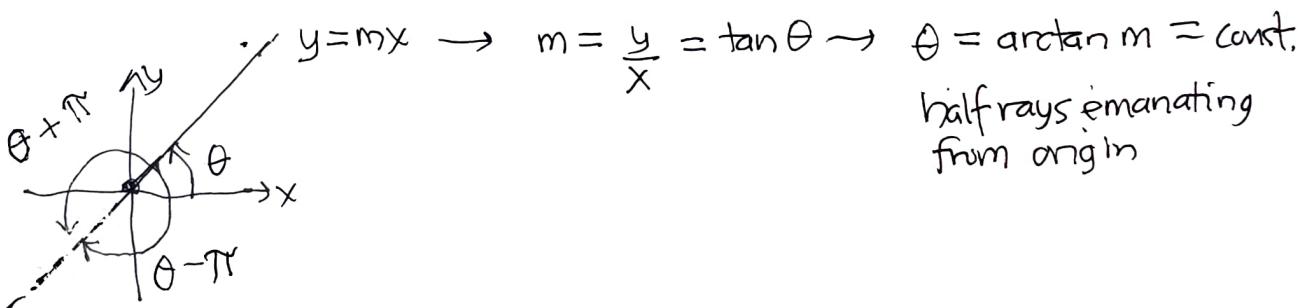
$\theta \approx -\infty \dots \infty$   
 traces out the line  
 from right to left



$\theta = \frac{\pi}{2} \dots \frac{3\pi}{2}$   
 traces out the line  
 from top to bottom

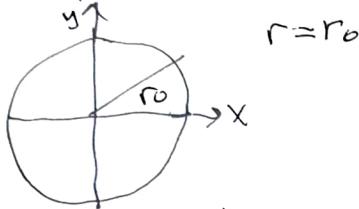


TOO COMPLICATED  
 to be useful  
 (ignore)



## Polar Coordinates : "grid" (4)

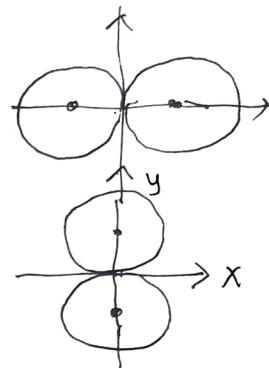
"simple" circles



centered at origin  $\rightarrow$  easy!

circles passing thru origin with center on an x-y axis :

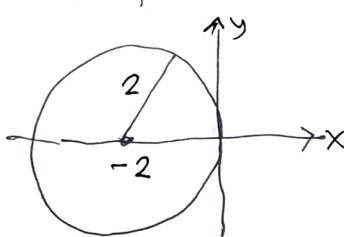
4 cases:



$$r = \pm a \cos \theta$$

$$r = \pm a \sin \theta$$

example:



$$(x+2)^2 + y^2 = 2^2 = 4$$

$$x^2 + 4x + 4 + y^2 = 4$$

$$\cancel{x^2} + \cancel{y^2} + 4x = 0$$

$$r^2 = 4(r \cos \theta)$$

$$r(r + 4 \cos \theta) = 0$$

$$r=0, \boxed{r = -4 \cos \theta}$$

go backwards  
if given this equation  
re-express in cartesian  
multiply by  $r$ :

$$\cancel{r^2} = -4 \cancel{r \cos \theta}$$

$$x^2 + y^2 = -4x$$

$$x^2 + \cancel{4x} + y^2 = 0$$

$$(x+2)^2 - 4$$

$$(x+2)^2 + y^2 = 4 = 2^2$$

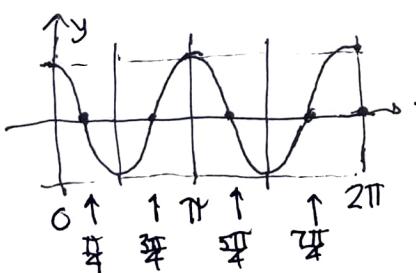
center:  $(-2, 0)$

radius: 2

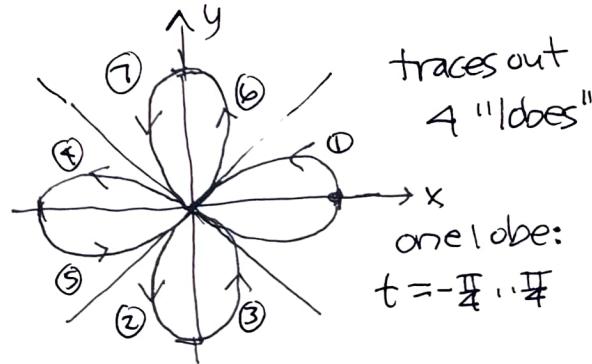
These are useful for  
toy math problems  
and some applications.

multiple loop curves

$$r = \cos 2\theta, \theta \in [0, 2\pi]$$



zeros at odd multiples  
of  $\pi/4$



traces out  
4 "lobes"

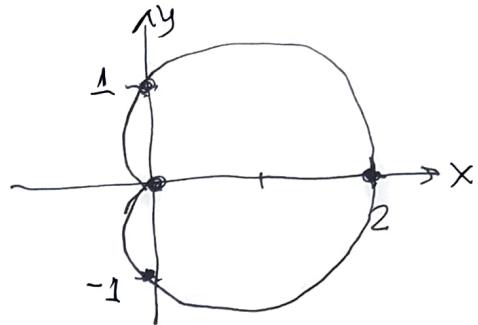
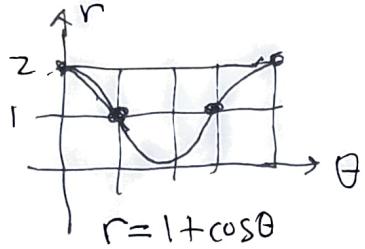
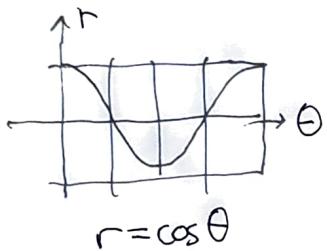
one lobe:

$$t = -\frac{\pi}{4} \dots \frac{\pi}{4}$$

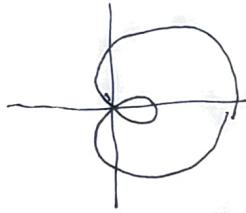
## Polar Coordinates: "grid" (5)

cardioids!  $r = a + b \cos \theta$ ,  $r = a + b \sin \theta$

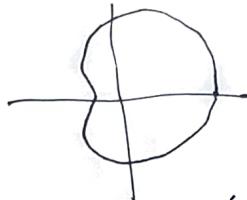
example:  $\boxed{r = 1 + \cos \theta} \rightarrow = 0 \text{ when } \cos \theta = -1 \rightarrow \theta = \pm \pi$   
 $\geq 0!$



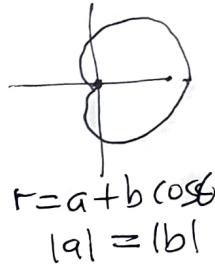
three cases:



$r = a + b \cos \theta$ ,  $|a| < |b|$   
 goes negative



$r = a + b \cos \theta$ ,  $|a| > |b|$   
 .. never negative



$r = a + b \cos \theta$   
 $|a| = |b|$