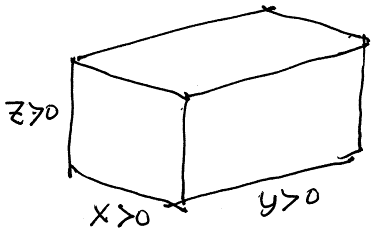
Max-Min Word Problem Example

A rectangular box with no lid is to be made from $12\,m^2$ of cardboard. Find the maximum volume of the box.

SOLUTION:



draw picture, introduce variable names (and ranges)

maximize volume $V = xyz > 0$
but it is function of 3 variables

variables are subject to the constraint of fixed area

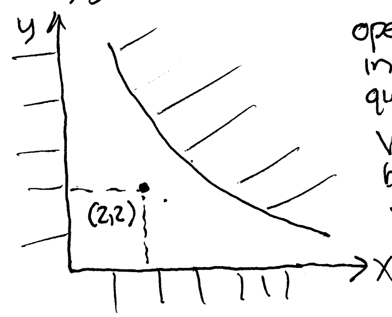
$$A = \underbrace{xy}_{\text{(bottom)}} + \underbrace{2xz}_{\text{(sides)}} + \underbrace{2yz}_{\text{(sides)}} = 12$$

use constraint to eliminate one variable (whichever convenient)
we pick z: $\quad xy + z(2)(x+y) = 12$

$$z = \frac{12 - xy}{2(x+y)} > 0$$

Backsub to get function of 2 variables

$$V = \frac{xy(12-xy)}{2(x+y)} = \frac{12xy - x^2y^2}{2(x+y)} = f(x,y) \text{ on } x>0,\ y>0 \text{ and } xy<12 \text{ so:}$$



open region in first quadrant
$V=0$ on boundary,
$V>0$ inside

V continuous function inside allowed region. must have a maximum.

↓

single critical pt $(2,2)$

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must be a local maximum, indeed the global maximum on these grounds

physical reasoning: clearly a largest such box must exist!

critical pts:

$$\frac{\partial V}{\partial x} = \cdots = \frac{y^2(12-2xy-x^2)}{2(x+y)^2} = 0$$

Maple

$$\frac{\partial V}{\partial y} = \cdots = \frac{x^2(12-2xy-y^2)}{2(x+y)^2} = 0$$

$x \neq 0, y \neq 0$

$$12 - 2xy - x^2 = 0 \longrightarrow 12 - 2x^2 - x^2 = 0$$
$$12 - 2xy - y^2 = 0 \qquad 3x^2 = 12$$
$$x^2 = 4$$

subtract: $y^2 - x^2 = 0 \rightarrow y = x \longrightarrow x = \pm 2 \rightarrow 2,\ y = 2$

no need to confirm with second derivative test!
but if we did we would find (let Maple do the work!)

$$\frac{\partial^2 V}{\partial x^2} = -\frac{y^2(y^2+12)}{(x+y)^3} < 0 \longrightarrow -\frac{4(16)}{4^3} = -1$$

$$\frac{\partial^2 V}{\partial y^2} = \frac{-x^2(x^2+12)}{(x+y)^3} < 0 \longrightarrow \quad \cdots \quad = -1$$

$$\frac{\partial^2 V}{\partial x \partial y} = -\frac{xy(x^2+3xy+y^2-12)}{(x+y)^3} \rightarrow -\frac{4(4)(1+3+1-3)}{4^3} = -\frac{1}{2}$$

$$V_{xx}V_{yy} - {V_{xy}}^2 = (-1)(-1) - (-\tfrac{1}{2})^2 > 0 \text{ confirmation}$$