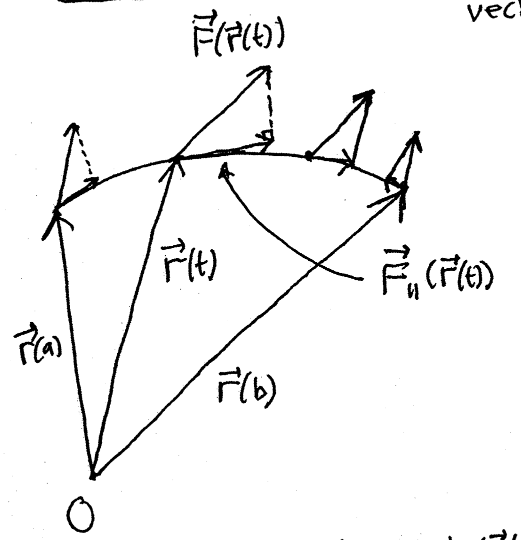
vector line integrals

$\vec{r} = \langle x, y, z \rangle$
vector variable



parametrized curve (directed: $|\vec{r}\,'(t)| \neq 0$)

$C: \vec{r} = \vec{r}(t),\ t = a..b$ (segment!)

unit normal $\hat{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ orients curve from $\vec{r}(a)$ to $\vec{r}(b)$

$\vec{F}(\vec{r})$ vector field

$\vec{F}(\vec{r}(t))$ vector field along C

$F_{\parallel}(\vec{r}(t)) = \vec{F}(\vec{r}(t)) \cdot \hat{T}(t)$ scalar component along C

$ds(t) = |\vec{r}\,'(t)|\,dt$ arclength differential

$$\vec{F}(\vec{r}(t)) \cdot \underbrace{\hat{T}(t)\, ds(t)}_{\frac{\vec{r}\,'(t)}{\cancel{|\vec{r}\,'(t)|}}\left(\cancel{|\vec{r}\,'(t)|}\,dt\right)} = \vec{F}(\vec{r}(t)) \cdot \underbrace{\vec{r}\,'(t)\,dt}_{d\vec{r}(t)}$$

"line integral of $\vec{F}$ along C is the integral of the scalar component $F_{\parallel}$ along C with respect to the differential of arclength"

$$\underbrace{\int_C \vec{F} \cdot \underbrace{\hat{T}\,ds}_{d\vec{r}}}_{=} = \boxed{\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \langle F_1, F_2, F_3 \rangle \cdot \langle dx, dy, dz \rangle$$
$$= \int_C \underbrace{F_1\,dx + F_2\,dy + F_3\,dz}_{\text{"inexact differential"}}$$

RECIPE

Given a parametrized curve C one simply evaluates the boxed formula.

Given an unparametrized "directed" curve C, find a parametrization $\vec{r} = \vec{r}(t)$ with the same direction and proceed as above

Given an integral

$$\int_C F_1\,dx + F_2\,dy + F_3\,dz$$

identify $\vec{F} = \langle F_1, F_2, F_3 \rangle$.

Then proceed as above.

Exercise

"unit" inverse square force pointing towards origin:

$$\vec{F} = \frac{-\hat{r}}{|\vec{r}|^2} = \frac{-\vec{r}}{|\vec{r}|^3} = \frac{-\langle x, y, z \rangle}{(x^2+y^2+z^2)^{3/2}}$$

C: straight line from $\langle 3, 4, 12 \rangle$ to $\langle 2, 2, 0 \rangle$

a) Find a parametrization for C as in Chapter 12.

b) Evaluate and simplify the integrand for the work done moving a body along C: $W = \int_C \vec{F} \cdot d\vec{r}$.

c) Evaluate the integral with Maple.

d) Use an obvious u-substitution to evaluate the same integral by hand.

e) Let $f(xyz) = (x^2+y^2+z^2)^{-1/2}$.
show that
$$W = f(\vec{r}(1)) - f(\vec{r}(0))$$
$$= f(2,2,0) - f(3,4,12)$$