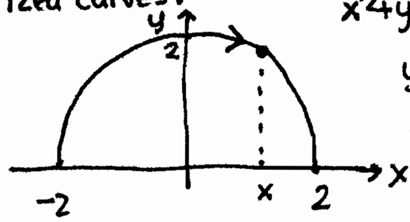
line integrals

parametrized curves:

C_1



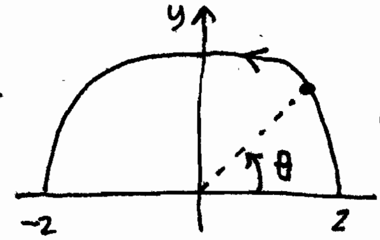
$x^2+y^2=4$

$\downarrow$

$y=\pm\sqrt{4-x^2}$

$\leftarrow \oplus$

C_2



$0\le\theta\le\pi$

$y=\sqrt{4-x^2}$ graph $\searrow$ $-2\le x\le 2$

$x=t \qquad t=-2..2$

$y=\sqrt{4-t^2}$

$\vec r=\langle t,\sqrt{4-t^2}\rangle$

$\vec r\,'=\langle 1,\frac{-t}{\sqrt{4-t^2}}\rangle$

$|\vec r\,'|=\sqrt{1+\frac{t^2}{4-t^2}}=\sqrt{\frac{4-t^2+t^2}{4-t^2}}=\frac{2}{\sqrt{4-t^2}}$

$ds=|\vec r\,'|dt=\frac{2dt}{\sqrt{4-t^2}}$

$\begin{matrix}x=r\cos\theta\\ y=r\sin\theta\end{matrix}\ \xrightarrow[\theta=t]{r=2}\ \begin{matrix}x=2\cos t\\ y=2\sin t\\ t=0..\pi\end{matrix}$

$\vec r=\langle 2\cos t,2\sin t\rangle$

$\vec r\,'=\langle -2\sin t,2\cos t\rangle=2\langle -\sin t,\cos t\rangle$

$|\vec r\,'|=2\sqrt{\sin^2t+\cos^2t}=2$

$ds=|\vec r\,'|dt=2\,dt$

line integral of scalars:

$\int_{C_1} y\,ds=\int_{-2}^{2}\cancel{(\sqrt{4-t^2})}\frac{2dt}{\cancel{\sqrt{4-t^2}}}=2t\Big|_{-2}^{2}=8$

$\int_{C_1} x\,ds=\int_{-2}^{2}\frac{\overbrace{t\,2dt}^{-du}}{\underbrace{\sqrt{4-t^2}}_{u}}=-2\sqrt{4-t^2}\Big|_{-2}^{2}=0$

$\int u^{-1/2}(-du)=-2u^{1/2}$

$\int_{C_1} 1\,ds=\int_{-2}^{2}\frac{2\,dt}{\sqrt{4-t^2}}\ \overset{\text{tech}}{\underset{\text{nology}}{=}}\ 2\arcsin\frac{t}{2}\Big|_{-2}^{2}$

$=2\arcsin 1-2\arcsin(-1)=2\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right]=2\pi$

center of gravity:

$\langle\bar x,\bar y\rangle=\frac{\langle\int_{C_1}x\,ds,\int_{C_1}y\,ds\rangle}{\int_{C_1}ds}$

$=\frac{\langle 0,8\rangle}{2\pi}=\langle 0,\frac{4}{\pi}\rangle$

≈ 1.27

(about 63% of 2)

$\int_{C_2} y\,ds=\int_0^{\pi}(2\sin t)(2dt)$

$=4(-\cos t)\Big|_0^{\pi}=8$

$\int_{C_2} x\,ds=\int_0^{\pi}(2\cos t)(2dt)$

$=4\sin t\Big|_0^{\pi}=0$

$\int_{C_2} 1\,ds=\int_0^{\pi}2\,dt=2t\Big|_0^{\pi}=2\pi$

$\langle\bar x,\bar y\rangle=\frac{\langle\int_{C_2}x\,ds,\int_{C_2}y\,ds\rangle}{\int_{C_2}ds}$

$=\frac{\langle 0,8\rangle}{2\pi}=\langle 0,\frac{4}{\pi}\rangle$

Note: C_1 and C_2 trace out the same path (set of points) but in opposite directions so symbolically $C_2=-C_1$.
This does not affect scalar line integrals.

line integrals of vectors

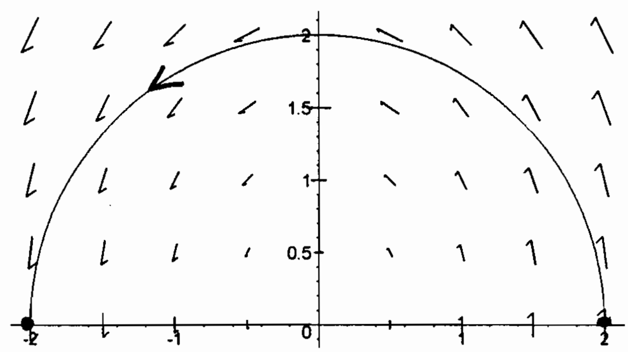
```
> with(plots)
Warning, the name changecoords has been redefined

> fieldplot([-y,2*x],x=-2..2,y=0..2,
  scaling=constrained,grid=[9,5],thickness=2):
  plot([2*cos(t),2*sin(t),t=0..Pi],thickness=2): display(%,%%);
```

$\vec{F} = \langle -y, 2x \rangle = \langle -y, 0 \rangle + \langle 0, 2x \rangle$

since $\vec{F}$ always makes a small acute angle with $\vec{r}\,'$ for C_2, its line integral will be positive, but for $C_1 = -C_2$, the result will change sign.

One can use C_1 to evaluate the line integral on C_2 by reversing its sign.

C_2



vector approach:

C_1 : $\begin{cases} \vec{r}\,'(t) = \langle 1, \frac{-t}{\sqrt{4-t^2}} \rangle \\ \vec{r}(t) = \langle t, \sqrt{4-t^2} \rangle \end{cases}$

$\vec{F}(\vec{r}(t)) = \langle -\sqrt{4-t^2}, 2t \rangle$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = -\sqrt{4-t^2} - \frac{2t^2}{\sqrt{4-t^2}}$

$= -\frac{(4-t^2+2t^2)}{\sqrt{4-t^2}} = -\frac{(4+t^2)}{\sqrt{4-t^2}}$

$\int_{C_1} \vec{F} \cdot \vec{r}\,' dt = -\int_{-2}^{2} \frac{4+t^2}{\sqrt{4-t^2}} dt$

$\frac{\text{tech}}{\text{nology}}$ $- 6 \arcsin \frac{t}{2} + \frac{t}{2}\sqrt{4-t^2} \Big|_{-2}^{2}$

$= -6(\arcsin 1 - \arcsin(-1))$

$= -6\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \boxed{-6\pi}$

C_2 : $\begin{cases} \vec{r}\,'(t) = \langle -2\sin t, 2\cos t \rangle \\ \vec{r}(t) = \langle 2\cos t, 2\sin t \rangle \end{cases}$

$\vec{F}(\vec{r}(t)) = \langle -2\sin t, 4\cos t \rangle$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = 4\sin^2 t + 8\underbrace{\cos^2 t}_{1-\sin^2 t} = 8 - 4\sin^2 t$

$\int_{C_2} \vec{F} \cdot \vec{r}\,' dt = \int_0^{\pi} 8 - 4\sin^2 t \, dt$

$\frac{\text{tech}}{\text{nology}}$ $4\left(2 - \frac{1}{2}(t - \sin t \cos t)\right]\Big|_0^{\pi}$

$= 4\left(\frac{3}{2}\right)\pi = \boxed{6\pi}$

$\longrightarrow \int_{-C_1} \vec{F} \cdot \vec{r}\,' dt = -\int_{C_1} \vec{F} \cdot \vec{r}\,' dt = \int_{C_2} \vec{F} \cdot \vec{r}\,' dt$

scalar component approach:

$\vec{F} \cdot d\vec{r} = \langle -y, 2x \rangle \cdot \langle dx, dy \rangle = -y\,dx + 2x\,dy$

C_1:

$x = t \qquad dx = dt$

$y = \sqrt{4-t^2} \qquad dy = \frac{-t}{\sqrt{4-t^2}} dt$

$\vec{F} \cdot d\vec{r} = -\left(\sqrt{4-t^2}\right)dt + 2(t)\left(\frac{-t\,dt}{\sqrt{4-t^2}}\right)$

$= \left(-\sqrt{4-t^2} - \frac{2t^2}{\sqrt{4-t^2}}\right) dt$

$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{-2}^{2} -\sqrt{4-t^2} - \frac{2t^2}{\sqrt{4-t^2}} \, dt$

$= \cdots = -6\pi$

C_2:

$x = 2\cos t \qquad dx = -2\sin t \, dt$

$y = 2\sin t \qquad dy = 2\cos t \, dt$

$\vec{F} \cdot d\vec{r} = -(2\sin t)(-2\sin t\,dt) + 2(2\cos t)(2\cos t\,dt)$

$= (4\sin^2 t + 8\cos^2 t)\,dt$

$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^{\pi} 4\sin^2 t + 8\cos^2 t \, dt$

$= \cdots = 6\pi$

Only the direction of a parametrized curve affects the value of the line integral, not the choice of parametrization itself.