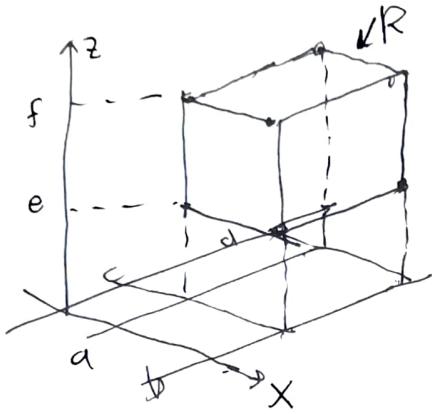


triple integrals (1)



R: rectangular box
 $[a,b] \times [c,d] \times [e,f]$

$\Delta V = \Delta x \Delta y \Delta z$

$\sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^p f(x_i^*, y_j^*, z_k^*) \Delta V$

$x_i^* = a + i \Delta x$
 $y_j^* = c + j \Delta y$
 $z_k^* \text{ in } k\text{th interval}$

$\Delta x = (b-a)/m$
 $\Delta y = (d-c)/n$
 $\Delta z = (f-e)/p$

Volume:
 $\Delta V = \Delta x \Delta y \Delta z$
 (x_i^*, y_j^*, z_k^*)
 in $i-j-k$ box

$$\begin{aligned} \iiint_R f(x, y, z) dV &= \lim_{m \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^p f(x_i^*, y_j^*, z_k^*) \Delta V \\ &= \left(\lim_{m \rightarrow \infty} \sum_{i=1}^m \left(\lim_{n \rightarrow \infty} \sum_{j=1}^n \underbrace{\left(\lim_{p \rightarrow \infty} \sum_{k=1}^p f(x_i^*, y_j^*, z_k^*) \Delta z \right)}_{\int_e^f f(x_i^*, y_j^*, z) dz} \Delta y \right) \Delta z \right) \Delta x \\ &= \underbrace{\int_a^b \left(\int_c^d \left(\int_e^f f(x, y, z) dz \right) dy \right) dx}_{\text{evaluate from inside out: nested integrals (order does not matter)}} \end{aligned}$$

evaluate from inside out:
 nested integrals
 (order does not matter)

Example R: $0 \leq x \leq 1, -1 \leq y \leq 2, 0 \leq z \leq 3$

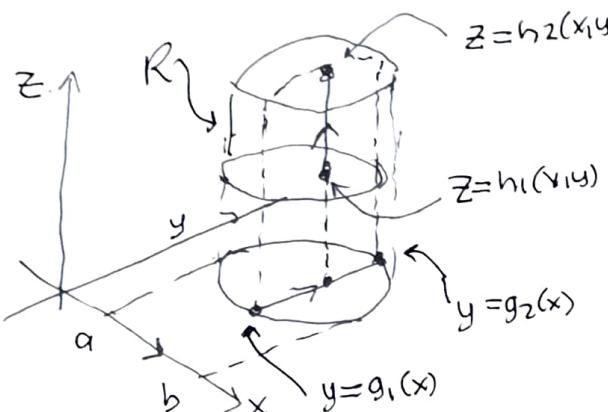
$$\begin{aligned} \iiint_R xyz^2 dV &= \int_0^1 \int_{-1}^2 \int_0^3 xyz^2 dz dy dx \\ &\quad \boxed{xyz^3/3 \Big|_{z=0}^{z=3} = 9xy} \\ &\quad \boxed{9xy \Big|_{y=-1}^{y=2} = \frac{9x}{2}(4-1) = \frac{27x}{2}} \\ &\quad \boxed{27x^2 \Big|_0^1 = \frac{27}{4}} \end{aligned}$$

details not important — easy to do

Triple integrals (2)

Nonconstant limits —

inner integral limits can depend on remaining variables not yet integrated over



$$z = h_1(x, y) \dots h_2(x, y) \\ \text{while } y = g_1(x) \dots g_2(x) \\ \text{while } x = a \dots b$$

enclose space between 2 surface graphs
above the area between 2 curve graphs
over an interval of the final axis

$$\iiint_R f(x, y, z) dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x, y)}^{h_2(x, y)} f(x, y, z) dz dy dx$$

{ sweeps out vertical line cross-section
sweeps line cross-section to vertical plane cross-section

sweeps vertical plane cross-section across solid region

Example

R is interior of sphere

$$x^2 + y^2 + z^2 = 1 \Leftrightarrow z = \pm \sqrt{1 - x^2 - y^2}$$

projects to $x^2 + y^2 = 1$ at $z = 0$ projects to $x = \pm 1$ when $y = 0$

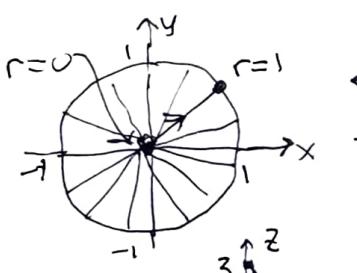
$$y = \pm \sqrt{1 - x^2}$$

$$z = -\sqrt{1 - x^2 - y^2}, \sqrt{1 - x^2 - y^2} \text{ while } y = -\sqrt{1 - x^2}, \sqrt{1 - x^2} \text{ while } x = -1, 1$$

$$\iiint_R f dV = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} f(x, y, z) dz dy dx$$

$$f(r \cos \theta, r \sin \theta, z) \equiv F(r, \theta, z)$$

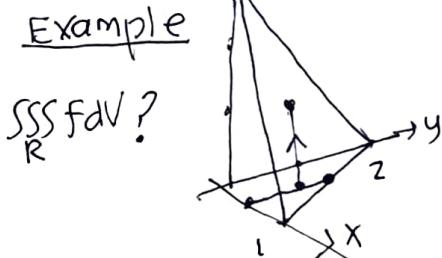
$$F(r, \theta, z) r dz dr d\theta$$



Symmetry
 $\rightarrow (r^2 + z^2 = 1)$
 $\rightarrow z = \pm \sqrt{1 - r^2}$

$\begin{cases} \text{ceiling} \\ \text{side walls} \\ \text{floor} \end{cases}$

Example



$$x + y + \frac{z}{3} = 1, \quad x = 0, y = 0, \quad z = 0 \\ \text{ceiling} \downarrow \\ z = -(x + y)/2 = 0 \\ \rightarrow x + y = 1 \rightarrow y = 2(1-x)$$

$$\boxed{\int_0^1 \int_0^{2(1-x)} \int_0^{3(1-x-y/2)} f dz dy dx}$$

Triple Integrals (3)

"reverse engineering" } a triple integral?
"deconstructing"

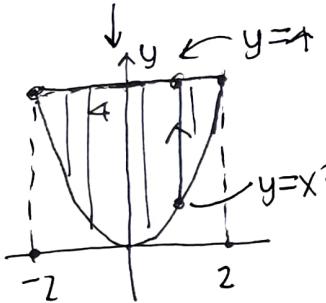
determine region of integration from limits of integration in order to re-order the iteration variable order.

$$Q = \int_{-2}^2 \int_{x^2}^4 \int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2} dz dy dx = \iiint_R \sqrt{x^2+z^2} dV, \text{ what is } R?$$

attach variables to limits to get 6 equations of bounding surfaces

$$\begin{cases} x=2 \\ y=4 \\ z=\sqrt{y-x^2} \\ x=-2 \\ y=x^2 \\ z=-\sqrt{y-x^2} \end{cases}$$

outer two



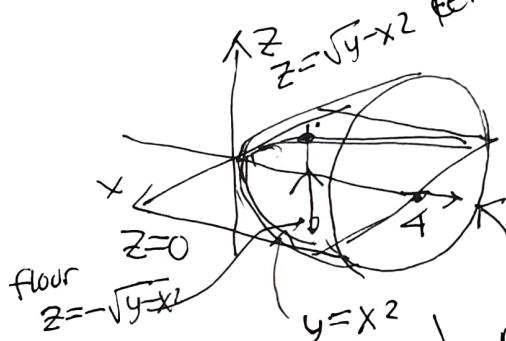
$$y = x^2 \dots 4$$

but top

$$\text{while } x = -2 \dots 2$$

"sidewalls":
when extended
in z-direction

but floor & ceiling meet at $z = 0$
so "walls" reduce to curves in xy plane



$$z^2 = y - x^2 \rightarrow x^2 + z^2 = y$$

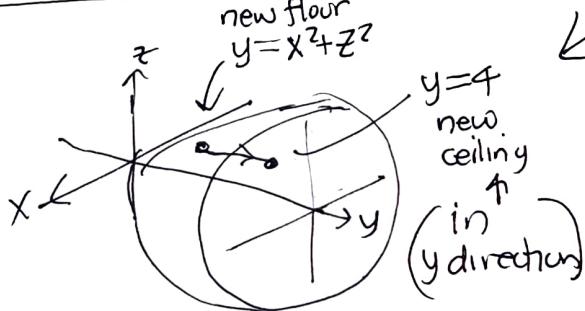
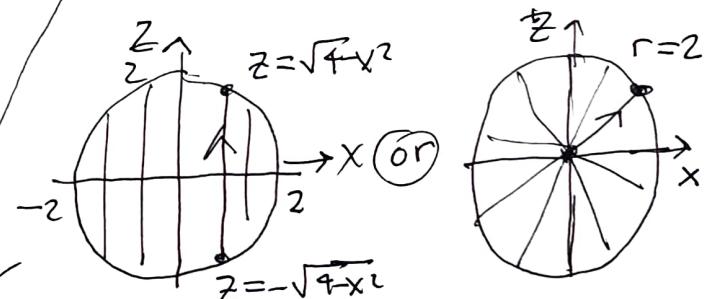
floor and ceiling "surfaces": ceiling paraboloid of revolution about y axis: $y = r^2 \equiv x^2 + z^2$

$$\begin{cases} x = r \cos \theta \\ z = r \sin \theta \end{cases}$$

polar coords in x-z plane

$$\text{project onto } \begin{cases} y = x^2 z^2 \\ x-z \text{ plane: } \begin{cases} y = 4 \\ x^2 + z^2 = 4 \end{cases} \end{cases}$$

'intersect'



$$\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \dots dz dx \left| \int_0^{2\pi} \int_0^2 \dots r dr d\theta \right|$$

$$\left. \int_{x^2+z^2}^4 \sqrt{x^2+z^2} dt \right| \left. \int_{r^2}^4 r \cdot dt \right|$$

$$Q = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+z^2}^4 \sqrt{x^2+z^2} dy dz dx =$$

$$\left. \int_0^{2\pi} \left\{ \int_0^2 \int_{r^2}^4 r^2 dy dr d\theta \right\} d\theta \right| \text{(try it by hand)}$$

triple integrals (4)

changing order of integration

some 3-d regions allow more than 1 integration order
maximum # re-orders is $3 \cdot 2 \cdot 1 = 3! = 6$ permutations
of x, y, z

z -first then a region of $x-y$ plane allowing either order

$$\iiint f \, dz \, dy \, dx$$

$\underbrace{\hspace{1cm}}$
 $\underbrace{\hspace{1cm}}$
dx or dy

y -first then a region of $x-z$ plane allowing either order

$$\iiint f \, dy \, dz \, dx$$

$\underbrace{\hspace{1cm}}$
 $\underbrace{\hspace{1cm}}$
dx or dz

x -first then a region of $y-z$ plane allowing either order

example: unit sphere: $x^2 + y^2 + z^2 = 1$

symmetric under all permutations of x, y, z

all 6 orders of iteration possible

wrote down one, then just permute variables!

[previous example did $dz \, dy \, dx$ order]

example: cut off first octant by a plane: $\frac{x}{2} + \frac{y}{4} + \frac{z}{6} = 1$

linear condition, all
variables on same footing.

6 orders of integration
possible

triple integrals (5)

Example

region bounded by
4 planes
"tetrahedron"

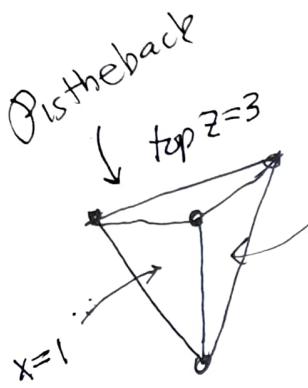
$$x=1, y=2, z=3, \underbrace{\frac{x}{2} + \frac{y}{4} + \frac{z}{6} = 1}_{\text{"f"}}$$

axis intercepts:

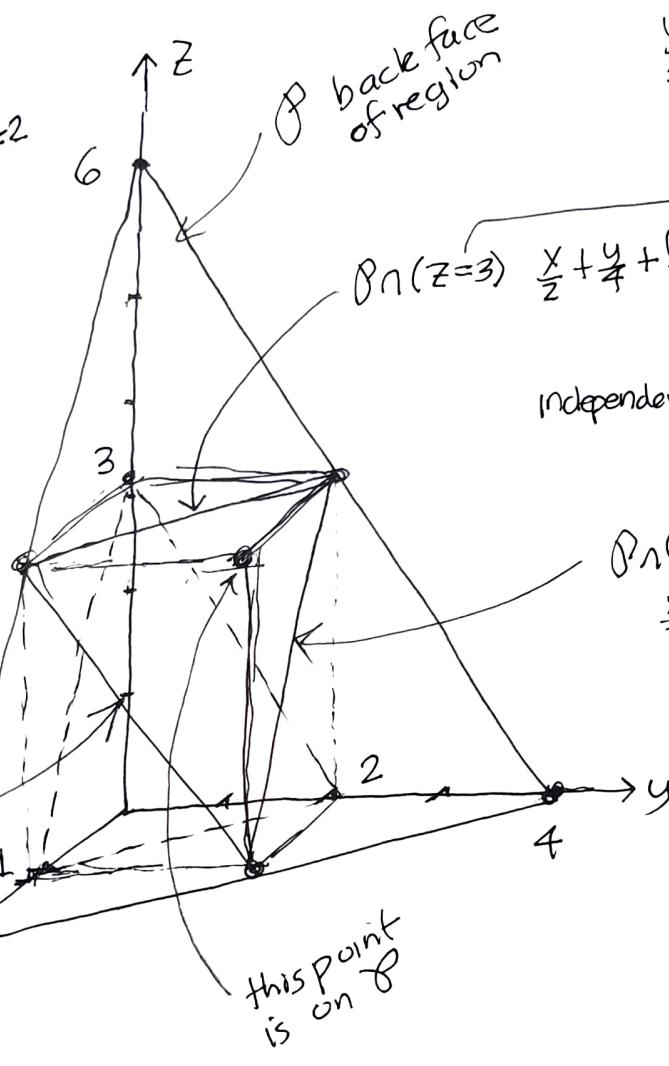
$$x=2 \quad (y=z=0)$$

$$y=4 \quad (x=z=0)$$

$$z=6 \quad (x=y=0)$$

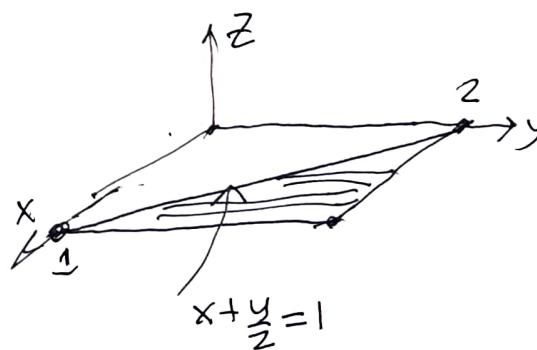
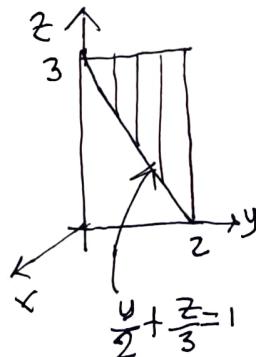
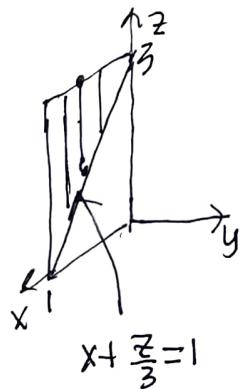


we need to
describe 3 oblique
edges to
project onto
the 3 coord
planes



$$\cap (x=1): \frac{1}{2} + \frac{y}{4} + \frac{z}{6} = 1 \rightarrow \frac{y}{4} + \frac{z}{6} = \frac{1}{2}$$

$$\frac{y}{2} + \frac{z}{3} = 1 \quad (\text{independent of } x \text{ projects onto } yz \text{ plane})$$



tetrahedron:

$$x=1, y=2, z=3$$

$$\frac{x}{2} + \frac{y}{4} + \frac{z}{6} = 1$$

solve for each variable in turn:

$$x = 2(1 - \frac{y}{4} - \frac{z}{6})$$

$$y = 4(1 - \frac{x}{2} - \frac{z}{6})$$

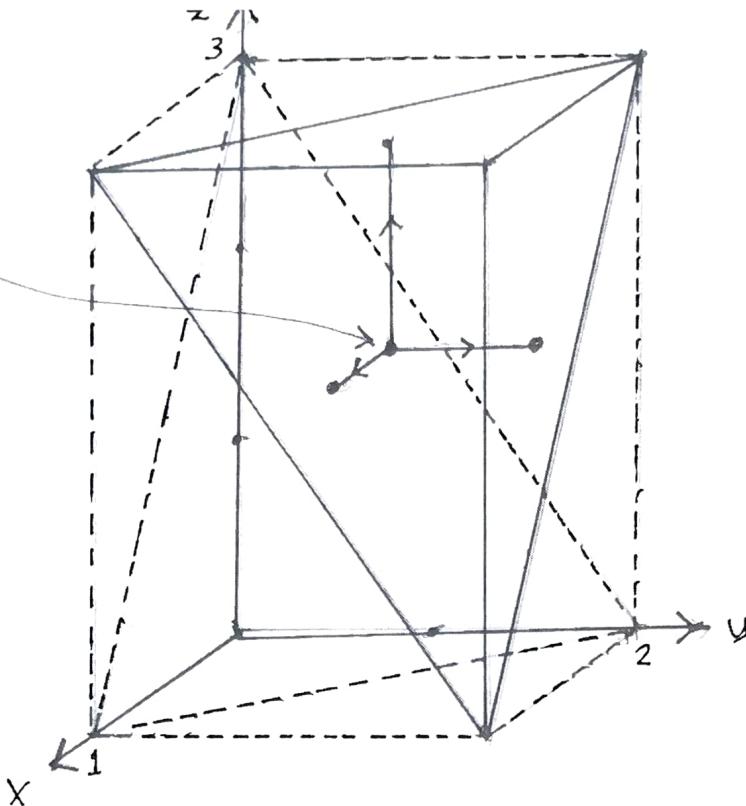
$$z = 6(1 - \frac{x}{2} - \frac{y}{4})$$

starting values for 3 partial integrations

ending values:

$$\begin{aligned} x &= 1 \\ y &= 2 \\ z &= 3 \end{aligned}$$

these are lower & upper limits for innermost integration



6 way iteration example

innermost integral moves along coded axes from the oblique plane outward from origin

outer double integral is done over projection of solid to coordinate planes of remaining variables

dashed triangles are those projections.

intersections of faces are common solutions of pairs of equations, eliminate one variable:

$$x=1 \rightarrow \frac{x}{2} + \frac{y}{4} + \frac{z}{6} = 1 \rightarrow \frac{1}{2} + \frac{y}{4} + \frac{z}{6} = 1 \rightarrow \frac{y}{2} + \frac{z}{3} = 1$$

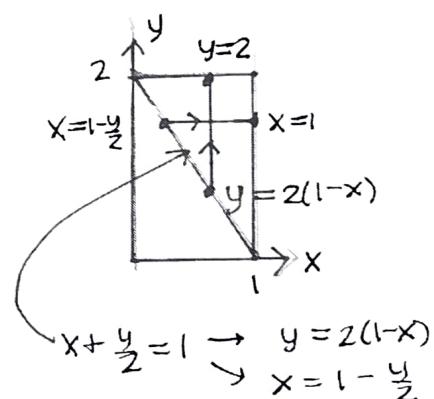
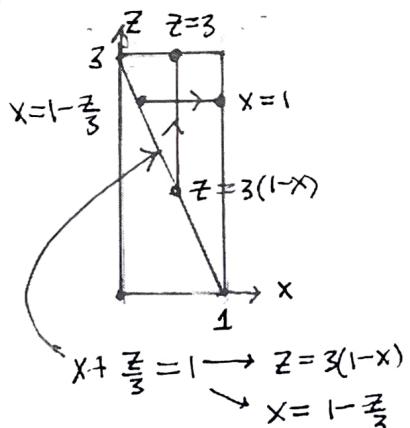
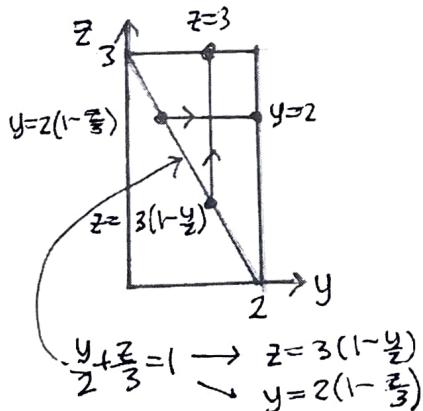
project solid onto YZ plane

$$y=2 \rightarrow \frac{x}{2} + \frac{y}{4} + \frac{z}{6} = 1 \rightarrow \frac{x}{2} + \frac{2}{4} + \frac{z}{6} = 1 \rightarrow x + \frac{z}{3} = 1$$

project solid onto XZ plane

$$z=3 \rightarrow \frac{x}{2} + \frac{y}{4} + \frac{z}{6} = 1 \rightarrow \frac{x}{2} + \frac{y}{4} + \frac{3}{6} = 1 \rightarrow x + \frac{y}{2} = 1$$

project solid onto XY plane



$$\left(\int_{2(1-\frac{y}{4}-\frac{z}{6})}^1 f dx \right)$$

$$\left(\int_{4(1-\frac{x}{2}-\frac{z}{6})}^2 f dy \right)$$

$$\left(\int_{6(1-\frac{x}{2}-\frac{y}{4})}^3 f dz \right)$$

$$\int_0^2 \int_{3(1-\frac{y}{2})}^3 (\quad) dz dy$$

$$\int_0^1 \int_{3(1-x)}^3 (\quad) dz dx$$

$$\int_0^1 \int_{2(1-x)}^2 (\quad) dy dx$$

$$\int_0^3 \int_{3(1-\frac{z}{6})}^2 (\quad) dy dz$$

$$\int_0^3 \int_{1-\frac{z}{3}}^1 (\quad) dx dz$$

$$\int_0^2 \int_{1-\frac{y}{2}}^1 (\quad) dx dy$$

when $f(xyz)=1$, all integrals give the volume: 1.