

## Scalar line integrals (1)

scalar integration so far :

$$1d : \int_a^b f(x) dx$$

$dx$  differential of arclength

$$dA = dx dy = r dr d\theta$$

$$2d : \iint_R f(x,y) dA$$

$dA$  differential of area

$$3d : \iiint_R f(x,y,z) dV$$

$dV$  differential of volume

$$dV = dx dy dz = r dz dr d\theta \\ = r^2 \sin\phi \, dr d\theta dz$$

### 2d, 3d parametrized curves

$$L = \int_C ds = \int_a^b \underbrace{|\vec{r}'(t)|}_{ds} dt$$

$ds$  differential of arclength

$$ds = \sqrt{dx^2 + dy^2}, \\ \sqrt{dx^2 + dy^2 + dz^2}$$

"region of integration" is a subset of the whole space of lower dimension  
in contrast with previous examples, but we are integrating a scalar only defined along the parametrized curve.

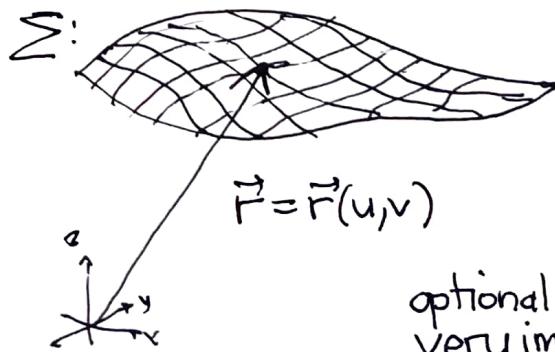
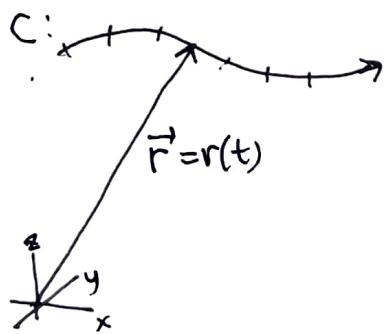
Functions ("scalar" or "fields") defined over the whole space can be integrated over subspaces, like curves in the plane, or curves and surfaces in space. Just parametrize the subspace and evaluate those functions on the parametrized position vector

$$\text{parametrized curves: } \vec{r} = \vec{r}(t) : f(\vec{r}) \rightarrow f(\vec{r}(t)) \quad \text{evaluate on curve}$$

$$\text{parametrized surfaces: } \vec{r} = \vec{r}(u,v) : f(\vec{r}) \rightarrow f(\vec{r}(u,v)) \quad \text{evaluate on surface}$$

then integrate over curve/surface with respect to:  
arclength  $ds = |\vec{r}'(t)| dt$  or

$$\text{surface area } \underbrace{dS}_{\substack{\text{geometric correction} \\ \text{factor to be discussed later}}} = (?) du dv$$

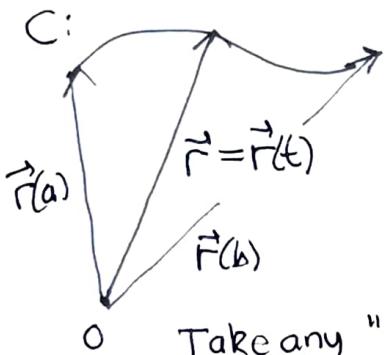


optional but very important!

## Scalar line integrals (2)

curves can be defined by a pair of equations in 3-d (intersection) of two surfaces or by a graph in 2-d

BUT we need a parametrization of the curve to evaluate a scalar line integral:  $\vec{r} = \vec{r}(t), a \leq t \leq b$  curve segment



Take any "scalar field"  $f(\vec{r})$   
 $\downarrow$   
 $f(\vec{r}(t))$

note arclength function:

$$s = \int_{t_0}^t |\vec{r}'(u)| du \Leftrightarrow \frac{ds}{dt} = |\vec{r}'(u)|$$

"dummy variable"

$$ds = |\vec{r}'(u)| du$$

geometric correction factor

$$\int_C f ds = \int_a^b f(\vec{r}(t)) |\vec{r}'(t)| dt$$

just plug in parametrization and integrate against differential of arclength

does not depend on parametrization!

$$\int_a^b f(\vec{r}(t)) \left| \frac{d\vec{r}(t)}{dt} \right| dt = \int_{t=a}^{t=b} f(\vec{r}(t(u))) \left| \frac{d\vec{r}(t(u))}{dt} \right| dt(u)$$

$\uparrow$   
change variable  
 $\frac{d\vec{r}(t(u))}{dt} = \frac{d\vec{r}(t(u))}{du} \frac{du}{dt}$   
 $\frac{d\vec{r}(t(u))}{dt} \frac{du}{dt} = \frac{d\vec{r}(t(u))}{du}$   
 $\frac{d\vec{r}(t(u))}{du} \frac{1}{dt} dt = \frac{d\vec{r}(t(u))}{du}$   
 $\frac{d\vec{r}(t(u))}{du} = \frac{d\vec{r}(t(u))}{du}$

$$= \int_{u_a}^{u_b} f(\vec{r}(t(u))) \left| \frac{d\vec{r}(t(u))}{du} \right| du$$

where  $t = t(u) \Leftrightarrow u = u(t)$   
 $u_a = u(a), u_b = u(b)$

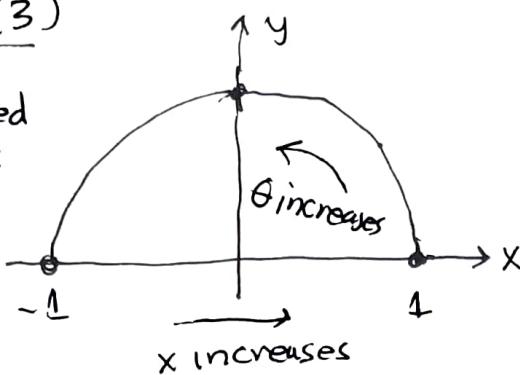
same formula for new parametrization:  $\vec{r} = \vec{r}(t(u))$

(notice absolute value sign here which we ignored in cancelling)

→ BUT  $u_b \geq u_a$  only if  $u = u(t)$  is an increasing function.  
 HOWEVER, taking abs val sign into account reorders  $u_a$  and  $u_b$  to be in the right order!  
 (sign of  $f$  determines sign of integral)

## Scalar line integrals (3)

Example: unparametrized curve in 2-d



unit semicircle:  
 $x^2 + y^2 = 1$ ,  
 $y \geq 0$

### Cartesian coord parametrization

$$x^2 + y^2 = 1 \rightarrow y = \sqrt{1-x^2} \text{ for } y \geq 0$$

$$x = -1 \dots 1$$

so "y =  $\sqrt{1-x^2}$  while x = -1 .. 1"

$$\vec{r} = \langle x, y \rangle = \langle x, \sqrt{1-x^2} \rangle$$

$$= \langle t, \sqrt{1-t^2} \rangle, t = -1..1$$

(choose  $t=x$  to be the parameter)

note increasing x moves in clockwise direction

oppositely directed  
parametrized  
curves

$$\vec{r}'(t) = \left\langle 1, \frac{1}{2} \left( \frac{-2t}{\sqrt{1-t^2}} \right) \right\rangle = \left\langle 1, -\frac{t}{\sqrt{1-t^2}} \right\rangle$$

$$|\vec{r}'(t)| = \sqrt{1 + \frac{t^2}{1-t^2}} = \sqrt{\frac{1-t^2+t^2}{1-t^2}} = \sqrt{\frac{1}{1-t^2}}$$

$$= \frac{1}{\sqrt{1-t^2}}$$

$$ds = \frac{dt}{\sqrt{1-t^2}}$$

$$\int_{C_{CW}} y \, ds = \int_{-1}^1 \left( \sqrt{1-t^2} \right) \frac{dt}{\sqrt{1-t^2}}$$

$$= \int_{-1}^1 1 \, dt = t \Big|_{-1}^1 = \boxed{2}$$

simple function  
to integrate.

### polar coord parametrization

$$1 = x^2 + y^2 = r^2 \rightarrow r = 1$$

so "r = 1 while  $\theta = 0.. \pi$ "

$$x = r \cos \theta = \cos \theta$$

$$y = r \sin \theta = \sin \theta$$

(keep  $\theta$  as parameter name, why not?)

$$\vec{r} = \langle x, y \rangle = \langle \cos \theta, \sin \theta \rangle$$

$$\theta = 0.. \pi$$

note increasing  $\theta$  moves in counterclockwise direction

BUT doesn't matter for scalar line integrals!

$$\vec{r}'(\theta) = \langle -\sin \theta, \cos \theta \rangle$$

$$|\vec{r}'(\theta)| = \sqrt{\sin^2 \theta + \cos^2 \theta} = 1$$

(indeed  $\theta$  is arclength on unit circle!)

$$ds = d\theta$$

$$\int_C y \, ds = \int_0^\pi \sin \theta \, d\theta$$

$$= -\cos \theta \Big|_0^\pi = \boxed{2}$$

## Scalar line integrals (4)

application: center of mass / centroid. (inhomogeneous wire)

given a linear mass density distribution on a curve:

$$dm = \rho(t) ds(t) \text{ differential of mass on } \vec{r} = \vec{r}(t)$$

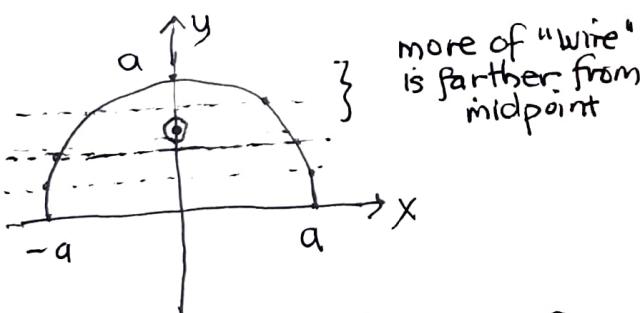
$$m = \int_C dm = \int_a^b \rho(t) ds(t) \quad \leftarrow ds(t) = |\vec{r}'(t)| dt$$

$$\langle M_y, M_x \rangle = \int_C \langle x, y \rangle dm = \int_a^b \langle x(t), y(t) \rangle \rho(t) ds(t)$$

$$\langle \bar{x}, \bar{y} \rangle = \frac{\langle M_y, M_x \rangle}{m} = \frac{\int_a^b \langle x(t), y(t) \rangle \rho(t) ds(t)}{\int_a^b \rho(t) ds(t)} = \int_a^b \langle x(t), y(t) \rangle \frac{\rho(t)}{m} ds(t)$$

If  $\rho(b) = \rho_0$  (homogeneous wire), then  $\rho$  cancels out!  
get centroid.

Example semicircular curve centroid



$$\vec{r}(\theta) = \langle a \cos \theta, a \sin \theta \rangle, \theta = 0.. \pi$$

$$|\vec{r}'(\theta)| = a, ds = a d\theta$$

$$L = \int_C ds = \pi a$$

$$\int_C \langle x, y \rangle ds = \langle 0, 2a^2 \rangle$$

$$\langle \bar{x}, \bar{y} \rangle = \frac{\langle 0, 2a^2 \rangle}{\pi a} = \langle 0, \frac{2a^2}{\pi} \rangle \approx \langle 0, 0.637 \rangle$$

a bit above the midpoint on the symmetry axis. ✓ makes sense!

variation:

suppose  $\rho \propto y$  (heavier farther from x axis)  
 $= ky, (k > 0, y \geq 0)$

$$= k(a \sin \theta)$$

$$\underbrace{\rho}_{ds}$$

$$\langle M, M_y, M_x \rangle = \int_0^\pi \langle 1, a \cos \theta, a \sin \theta \rangle (k a \sin \theta) (a d\theta)$$

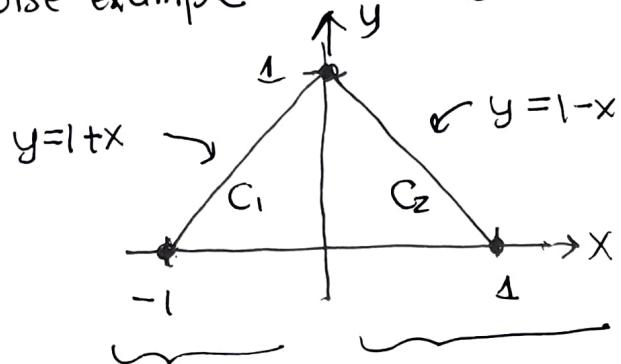
$$= k a \langle 2, 0, \frac{\pi a}{2} \rangle \rightarrow \bar{y} = \frac{\pi a / 2}{2} = \frac{\pi a}{4} \approx 0.785 a \text{ (higher as it should be)}$$

fractional  
mass distribution

## Scalar line integrals (5)

piecewise example

$$C = C_1 \cup C_2$$



$$x = t = -1..0$$

$$y = 1 + t$$

$$\vec{r} = \langle t, 1+t \rangle$$

$$\vec{r}' = \langle 1, 1 \rangle$$

$$|\vec{r}'| = \sqrt{1+1} = \sqrt{2}$$

$$x = t = 0..1$$

$$y = 1 - t$$

$$\vec{r} = \langle t, 1-t \rangle$$

$$\vec{r}' = \langle 1, -1 \rangle$$

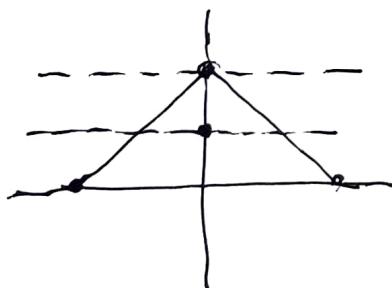
$$|\vec{r}'| = \sqrt{1+1} = \sqrt{2} \rightarrow ds = \sqrt{2} dt$$

$$\int_C \langle 1, y \rangle ds = \int_{C_1} \langle 1, y \rangle ds + \int_{C_2} \langle 1, y \rangle ds$$

$$= \int_{-1}^0 \langle 1, 1+t \rangle (\sqrt{2} dt) + \int_0^1 \langle 1, 1-t \rangle (\sqrt{2} dt)$$

$$= \dots = \langle \sqrt{2}, \sqrt{2}(1-\frac{1}{2}) \rangle + \langle \sqrt{2}, \sqrt{2}(1-\frac{1}{2}) \rangle$$

$$\stackrel{\text{easy by hand}}{=} \langle 2\sqrt{2}, \sqrt{2} \rangle \rightarrow \bar{y} = \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2} \text{ midpoint!}$$



each  $\Delta S$  has same moment about middle line!