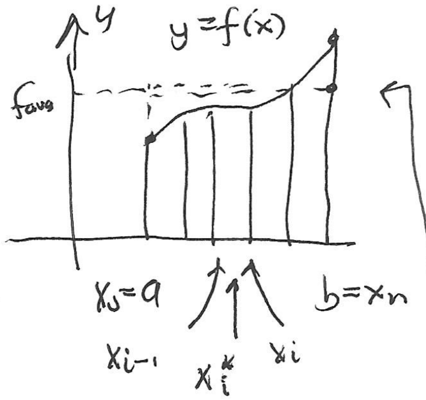
Riemann and average value of a function

1d:



$\Delta x = \frac{b-a}{n}$

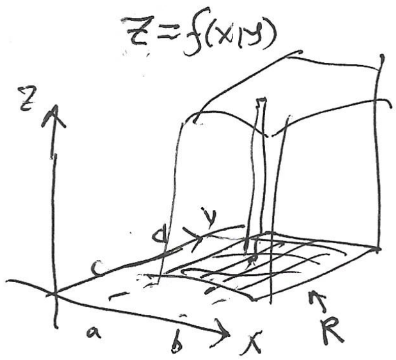
Interpretation: replace f by f_{avg} get same integral over $[a,b]$. rectangle has same area

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \underbrace{\Delta x}_{=\frac{b-a}{n}}$$

$$= (b-a) \underbrace{\lim_{n\to\infty} \underbrace{\left(\frac{\sum_{i=1}^{n} f(x_i^*)}{n} \right)}_{\text{average of sampled values.}}}_{\equiv f_{avg} \text{ over interval}}$$

$$f_{avg} = \underbrace{\frac{1}{b-a}}_{\text{divide integral by length of interval of integration}} \int_a^b f(x)\,dx$$

2d



$R: [a,b] \times [c,d]$

$\Delta x = \frac{b-a}{n}$, $\Delta y = \frac{d-c}{m}$

$A = (b-a)(d-c)$

$= \text{area}(R)$

$= \iint_R 1\,dA$

$$\iint_R f(x,y)\,dA$$

$$= \lim_{n\to\infty}\lim_{m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i^*, y_j^*) \underbrace{\Delta x \Delta y}_{\left(\frac{b-a}{n}\right)\left(\frac{d-c}{m}\right) = \frac{A}{nm}}$$

$$= \underbrace{\lim_{n\to\infty}\lim_{m\to\infty} \frac{1}{nm} \underbrace{\left(\sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i^*, y_j^*) \right)}_{\text{avg of sampled values}}}_{\equiv f_{avg}} \cdot A$$

number of sampled values

$$\boxed{f_{avg} = \frac{1}{A} \iint_R f(x,y)\,dA = \frac{\iint_R f(x,y)\,dA}{\iint_R 1\,dA}}$$

divide integral by area of rectangle of intergration

(in 3d divide by volume of rectangular box of integration)

Interpretation: replace f by f_{avg}, get same integral over rectangle.

$f_{min} \le f_{avg} \le f_{max}$ always!