integration over 2D and 3D regions of the plane and space

Setting up iterated double or triple integrals is all about parametrizing these regions by giving a set of nested functional relationships which specifying the starting and stopping values of the coordinates

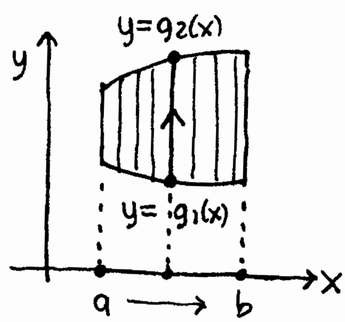
coords $\underbrace{\underbrace{(u,v}_{2D},w)}_{3D}$

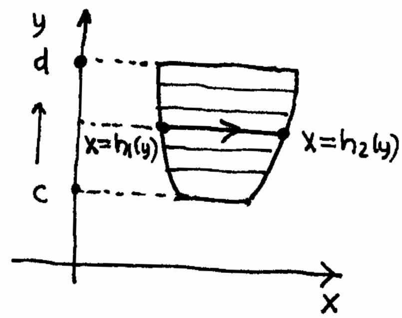
starting/stopping values:

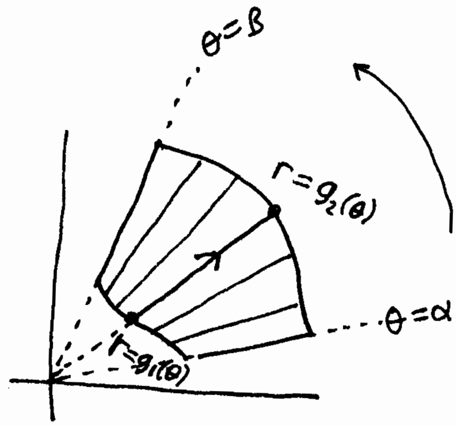
$$\left.\begin{array}{l} w = h_1(u,v) \,..\, h_2(u,v) \\ 2D \left\{ \begin{array}{l} v = g_1(u) \,..\, g_2(u) \\ u = a \,..\, b \end{array}\right. \end{array}\right\} 3D$$

2D regions R:

$$\iint_R f(x,y)\,dA$$







$y = g_1(x) \,..\, g_2(x)$

$x = a \,..\, b$

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, \underbrace{dy\,dx}_{dA}$$

$x = h_1(y) \,..\, h_2(y)$

$y = c \,..\, d$

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, \underbrace{dx\,dy}_{dA}$$

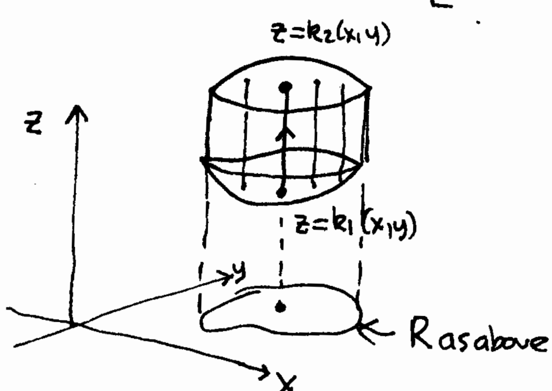
$r = g_1(\theta) \,..\, g_2(\theta)$

$\theta = \alpha \,..\, \beta$

$$\int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\, \underbrace{r\,dr\,d\theta}_{dA}$$

3D regions E:

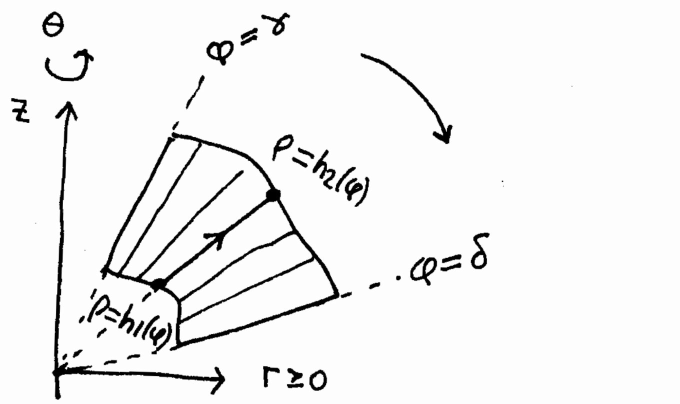
$$\iiint_E f(x,y,z)\,dV$$



$z = k_1(x,y) \,..\, k_2(x,y)$

$$\iint_R \int_{k_1(x,y)}^{k_2(x,y)} f(x,y,z)\,dz\,dA$$

(includes cylindrical coords when R is a polar region)



$\rho = h_1(\varphi) \,..\, h_2(\varphi)$

$\varphi = \gamma \,..\, \delta$

$\theta = \alpha \,..\, \beta$

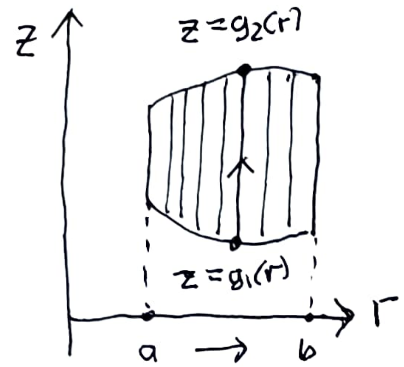
$$\int_\alpha^\beta \int_\gamma^\delta \int_{h_1(\varphi)}^{h_2(\varphi)} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\, \underbrace{\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta}_{dV}$$

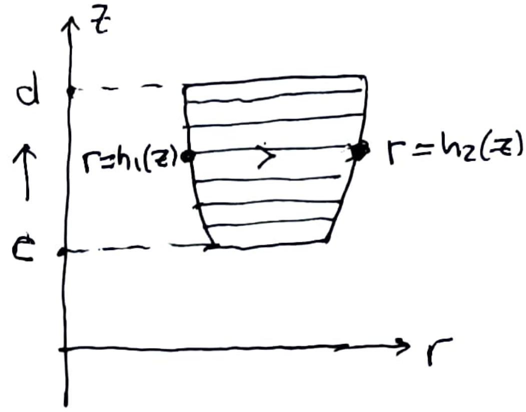
(this is special case of regions in r-z half-plane which revolve in the θ direction, i.e., about z-axis)

Integration over 2D and 3D regions of the plane and space (2)

In space in cylindrical coordinates for regions which are rotationally symmetric about the z-axis (including partial revolutions) an r-z half plane diagram is sufficient to describe the innermost integral of a triple integral $\iiint \cdots \, dr\, dz\, d\theta$ or $\iiint \cdots \, dz\, dr\, d\theta$

The diagrams for the x-y plane in Cartesian coordinates only have to be relabeled.





$z = g_1(r) \ldots g_2(r)$

$r = a \ldots b$

$$\int_{\theta_1}^{\theta_2} \int_a^b \int_{g_1(r)}^{g_2(r)} f \, r \, dz\, dr\, d\theta$$

$r = h_1(z) \ldots h_2(z)$

$z = c \ldots d$

$$\int_{\theta_1}^{\theta_2} \int_c^d \int_{h_1(z)}^{h_2(z)} f \, r \, dr\, dz\, d\theta$$