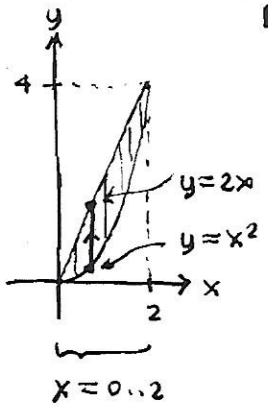
double integrals : it's really about describing a region of the plane



■ Consider the region R between the graphs $y=x^2$ and $y=2x$.
These curves intersect at : $x^2=2x$ or $0=x^2-2x=x(x-2) \to x=0,2$
$\to y=0,4$
points $(0,0)$ and $(2,4)$.

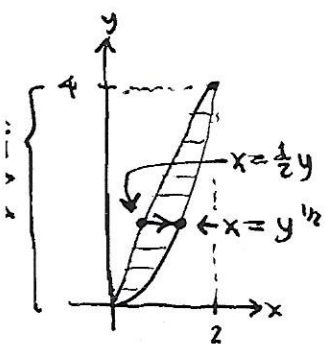
The diagram shows a "typical" vertical cross-section and its "direction" (increasing y) with endpoints labeled by start/stop values

R : $y=x^2..2x$ (increasing bot. to top)
as $x=0..2$ (increasing from left to right)

This describes the region as a "type I" region (y first, then x)

$$\iint_R f(x,y)\,dA = \int_{x=0}^{x=2}\int_{y=x^2}^{y=2x} f(x,y)\,dy\,dx = \int_0^2\int_{x^2}^{2x} f(x,y)\,dy\,dx$$

$y=x^2..2x$ as $x=0..2$



■ But we can also describe the bounding curves with x expressed as a function of y, a "type II" region (x first, then y)

$y=x^2 \xrightarrow{x\ge 0} x=y^{1/2}$

$y=2x \longrightarrow x=\frac{1}{2}y$

R : $x=\frac{1}{2}y..y^{1/2}$ (increasing left to right)
as $y=0..4$ (increasing bot. to top)

The diagram shows a "typical" horizontal cross-section and its "direction" (increasing x) with endpoints labeled by start/stop values

$$\iint_R f(x,y)\,dA = \int_{y=0}^{y=4}\int_{x=\frac{1}{2}y}^{x=y^{1/2}} f(x,y)\,dx\,dy = \int_0^4\int_{\frac{1}{2}y}^{y^{1/2}} f(x,y)\,dx\,dy$$

$x=\frac{1}{2}y..y^{1/2}$ as $y=0..4$

technique : changing the order of integration

For a region that allows either choice above, we can start with one order of integration, make a diagram of the region of integration and its bounding curves, then re-express them as above and determine the new limits of integration

$$\int_0^2\int_{x^2}^{2x} \underbrace{f(x,y)}_{z=f(x,y)}\,dy\,dx = \int_{x=0}^{x=2}\int_{y=x^2}^{y=2x} f(x,y)\,dy\,dx \longrightarrow \text{make diagram} \to \text{re-describe}$$

then go backwards to the new double integral

REMARK

$z=f(x,y)$ helps visualize integral in 3D as signed volume between graph and xy plane.

integrand is function whose graph in 3D leads to solid associated with 2D integral