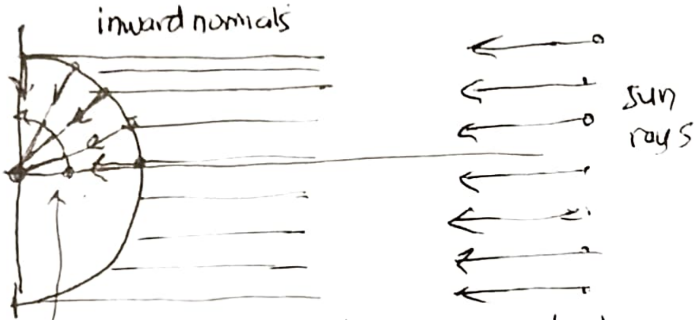
Flux in 2-d

consider the half of the Earth facing the sun — solid of revolution about symmetry axis connecting the centers of the Earth and sun

consider a plane cross-section through that symmetry axis



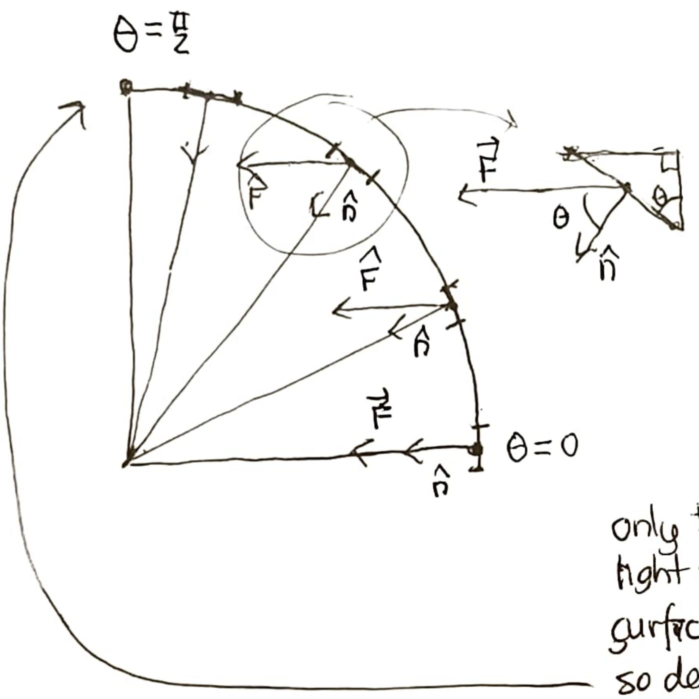
θ is angle with horizontal

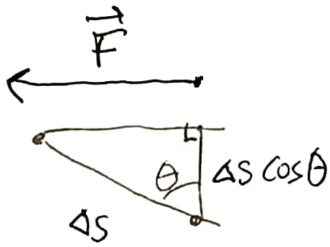
radius R

constant horizontal vector field

$\vec{F}$ is flux of energy hitting a line perpendicular to the rays per unit length

as we approach $\theta = \pi/2$, equally spaced increments in θ (hence of $\Delta S = R\Delta\theta$) receive less and less energy because of the "surface" tilting with respect to the rays

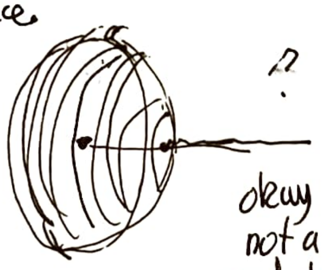




$$|\vec{F}|(\Delta S \cos\theta) = \vec{F}\cdot\hat{n}\,\Delta S = F_\perp\,\Delta S$$

only the normal component of the impinging light rays is effective in delivering energy to the surface — at the tips the rays are parallel & so deliver no energy: $\cos\theta \to 0$ there

Of course this revolves around the axis to a 3-d problem of annular rings on the Earth's surface.



okay not a good sketch

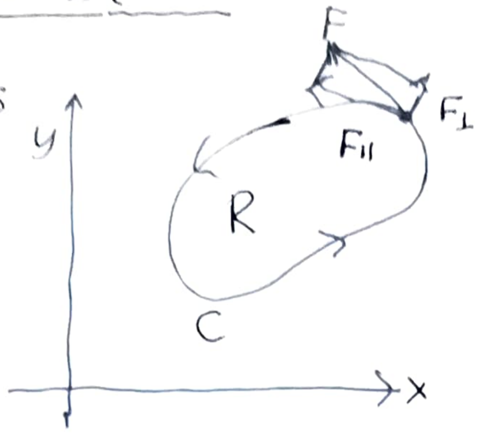
energy per unit area perpendicular to sun rays

$|\vec{F}|(\Delta A)_\perp$ is amount of energy delivered to any amount of surface area

integrating up the normal component (2-d or 3-d) against the arclength/area of the curve/surface measures the "flux" of that vector field through that curve/surface.

Green-Stokes-Gauss (& flux)

2-d Greens



$F_\perp = \hat{n} \cdot \vec{F}$

$F_\parallel = \hat{T} \cdot \vec{F}$

$\hat{n}$ = outward normal

$\hat{T}$ = counterclockwise unit normal

$\hat{n} \times \hat{T} = \hat{k}$

righthand rule!

Stokes version:

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C F_\parallel \, ds$$

circulation of $\vec{F}$ around C

$$= \iint_R \underbrace{(\text{curl}\vec{F}) \cdot \hat{k}}_{\text{"circulation density"}} \, dA$$

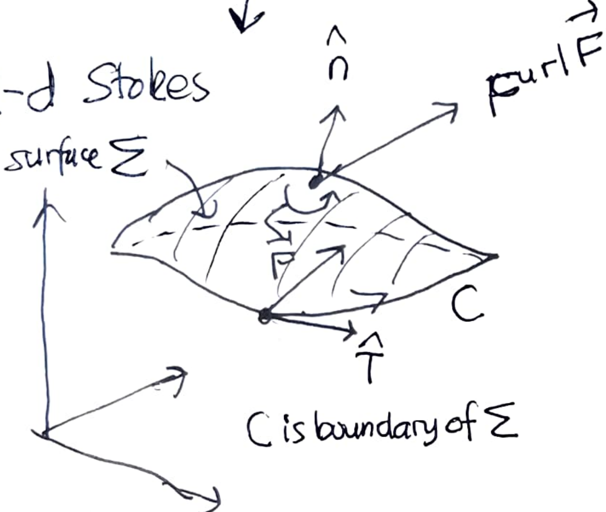
Gauss version:

$$\oint_C F_\perp \, ds$$

"flux" of $\vec{F}$ out of R thru C

$$= \iint_R \underbrace{(\text{div}\vec{F})}_{\text{"flux density"}} \, dA$$

3-d Stokes



C is boundary of Σ

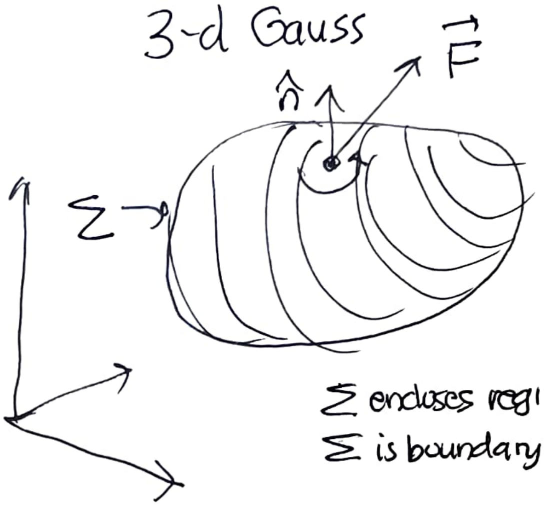
$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C F_\parallel \, ds$$

$$= \iint_\Sigma (\text{curl}\vec{F}) \cdot \underbrace{\hat{n} \, dS}_{\vec{dS}}$$

(flux of curl F thru Σ in direction $\hat{n}$)

$\vec{dS}$ vector differential of surface area

3-d Gauss



Σ encloses region R

Σ is boundary of R

$$\iint_\Sigma \vec{F} \cdot \vec{dS} = \iint_\Sigma F_\perp \, dS$$

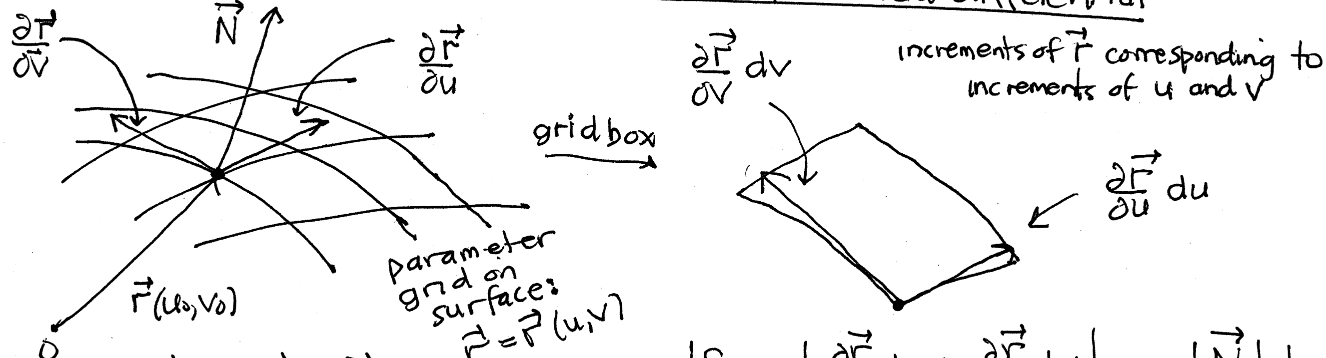
(flux of F thru Σ in direction $\hat{n}$)

$$= \iiint_R (\text{div}\vec{F}) \, dV$$

in both cases:

integral of vector field on boundary = integral of its appropriate derivative on the interior region inside the boundary

surface integral : normal vector and surface area differential



$$\vec{N} = \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$$

$$\hat{N} = \frac{\vec{N}}{|\vec{N}|}$$

$$dS = \left| \frac{\partial \vec{r}}{\partial u} du \times \frac{\partial \vec{r}}{\partial v} dv \right| = \underbrace{|\vec{N}|}_{\sqrt{\vec{N}\cdot\vec{N}}} du\,dv$$

cross-product gives area of parallelogram they determine.

typically complicated to integrate, like arclength!

surface integrals integrate the normal component of a vector field against the differential of surface area.

$$\iint_{\Sigma} \underbrace{\vec{F}(\vec{r}(u,v)) \cdot \hat{N}(\vec{r}(u,v)) \underbrace{dS}_{|\vec{N}(\vec{r}(u,v))|\,du\,dv}}_{\vec{N}(\vec{r}(u,v))}$$

surface

$$= \int_{v_1}^{v_2}\int_{u_1}^{u_2} \vec{F}(\vec{r}(u,v)) \cdot \vec{N}(\vec{r}(u,v))\, du\,dv$$

square root goes away for vector integrals

easier than integrating scalars

$$\left[\int_C \underbrace{\vec{F}(\vec{r}(t)) \cdot \hat{T}(\vec{r}(t)) \underbrace{ds}_{|\vec{r}\,'(t)|\,dt}}_{\vec{r}\,'(t)} \right.$$

$$= \int_{t_1}^{t_2} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

just like for the line integral case $\Big]$

Scalar surface integrals

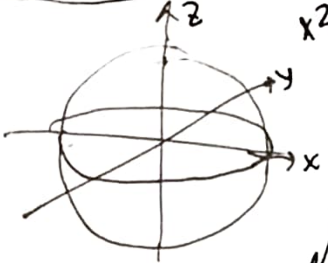
Like scalar line integrals, sqrt factor makes them more difficult to evaluate

$$\iint_{\Sigma} f\,dS = \int_{v_1}^{v_2}\int_{u_1}^{u_2} f(\vec{r}(u,v))\,\underbrace{|\vec{N}(\vec{r}(u,v))|}_{\left|\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v}\right| = \sqrt{\cdots}}\,du\,dv$$

sqrt factor

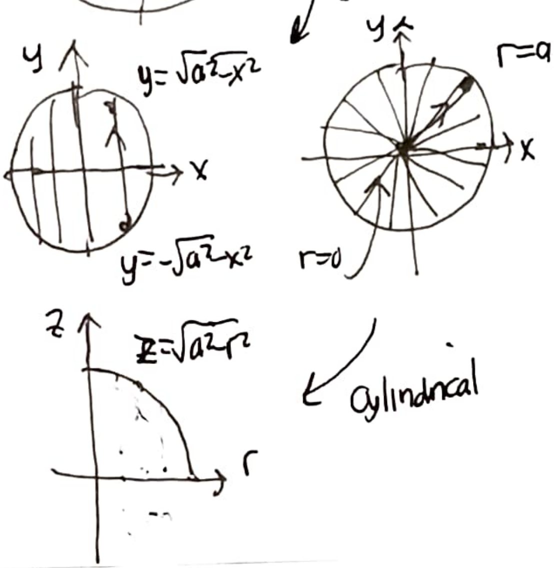
surface area of hemisphere $= 2\pi a^2 = \frac{1}{2}(4\pi a^2) = S$

cartesian coords



$\left.\begin{array}{l} x^2+y^2+z^2=a^2 \\ z\geq 0 \end{array}\right]$ $z=\sqrt{a^2-x^2-y^2}$ $\quad \vec{r}(x,y) = \langle x, y, \sqrt{a^2-x^2-y^2}\rangle$

$z=0$



$$\frac{\partial \vec{r}}{\partial x} = \langle 1, 0, \frac{1}{2}\frac{(-2x)}{\sqrt{a^2-x^2-y^2}}\rangle = \langle 1, 0, \frac{-x}{z}\rangle$$

$$\frac{\partial \vec{r}}{\partial y} = \langle 0, 1, -\frac{y}{z}\rangle$$

$$\frac{\partial \vec{r}}{\partial x}\times\frac{\partial \vec{r}}{\partial y} = \cdots = \langle \frac{x}{z}, \frac{y}{z}, 1\rangle = \frac{\langle x,y,z\rangle}{z} = \frac{\vec{r}}{z}$$

radially outward!

$$\left|\frac{\partial \vec{r}}{\partial x}\times\frac{\partial \vec{r}}{\partial y}\right| = \frac{\sqrt{x^2+y^2+z^2}}{z} = \frac{a}{z} = \frac{a}{\sqrt{a^2-x^2-y^2}}$$

$$S = \underbrace{\int_{-a}^{a}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}}}_{\int_0^{2\pi}\int_0^a} \underbrace{\frac{a}{\sqrt{a^2-x^2-y^2}}}_{\underbrace{\frac{a}{\sqrt{a^2-r^2}}}_{u\text{-sub!}}} \underbrace{dy\,dx}_{r\,dr\,d\theta}$$

hard

easier

$= 2\pi a^2$ ✓

spherical coords

$\vec{r} = a\langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle = a\hat{r}$

$\partial\vec{r}/\partial\phi = a\langle \cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi\rangle$

$\partial\vec{r}/\partial\theta = a\langle -\sin\phi\sin\theta, \sin\phi\cos\theta, 0\rangle$

$\frac{\partial \vec{r}}{\partial \phi}\times\frac{\partial \vec{r}}{\partial \theta} = \cdots = a^2\sin\phi\langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle = a^2\sin\phi\,\hat{r}$ (radial outward)

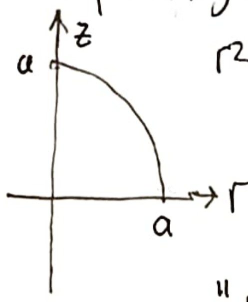
$\left|\frac{\partial \vec{r}}{\partial \phi}\times\frac{\partial \vec{r}}{\partial \theta}\right| = a^2\sin\phi$

$$S = \int_0^{2\pi}\int_0^{\pi/2} a^2\sin\phi\,d\phi\,d\theta = 2\pi a^2 \underbrace{\int_0^{\frac{\pi}{2}}\sin\phi\,d\phi}_{-\cos\phi|_0^{\pi/2} = 1} = 2\pi a^2$$

easiest!

Scalar surface integrals (2)

hemisphere: cylindrical coords $\quad \vec{r} = \langle r\cos\theta,\ r\sin\theta,\ \sqrt{a^2-r^2}\rangle,\ r=0..a,$
$\theta = 0..2\pi$



$r^2+z^2=a^2$
$z=\sqrt{a^2-r^2}$

$$\frac{\partial\vec{r}}{\partial r} = \left\langle \cos\theta, \sin\theta, \frac{1}{2}\frac{(-2r)}{\sqrt{a^2-r^2}} \right\rangle = \langle \cos\theta, \sin\theta, -r/z \rangle$$

$$\frac{\partial\vec{r}}{\partial\theta} = \langle -r\sin\theta, r\cos\theta, 0\rangle \equiv \langle -rs, rc, 0\rangle$$

"radial": $\frac{\partial\vec{r}}{\partial r}\times\frac{\partial\vec{r}}{\partial\theta} = \cdots = r\left\langle \frac{r}{z}c, \frac{r}{z}s, 1\right\rangle = \frac{r}{z}\underbrace{\langle r\cos\theta, r\sin\theta, z\rangle}_{\vec{r}}$

$$\left|\frac{\partial\vec{r}}{\partial r}\times\frac{\partial\vec{r}}{\partial\theta}\right| = r\sqrt{\frac{r^2}{a^2-r^2}\underbrace{(c^2+s^2)}_{1}+1} = r\sqrt{\frac{r^2+(a^2-r^2)}{a^2-r^2}} = \left(\frac{a}{\sqrt{a^2-r^2}}\right)r$$

$$dS = \left(\frac{a}{\sqrt{a^2-r^2}}r\right)dr\,d\theta$$

comes from $|\vec{N}|$, automatically

↓

same as cartesian integral re-expressed in cylindrical coords.