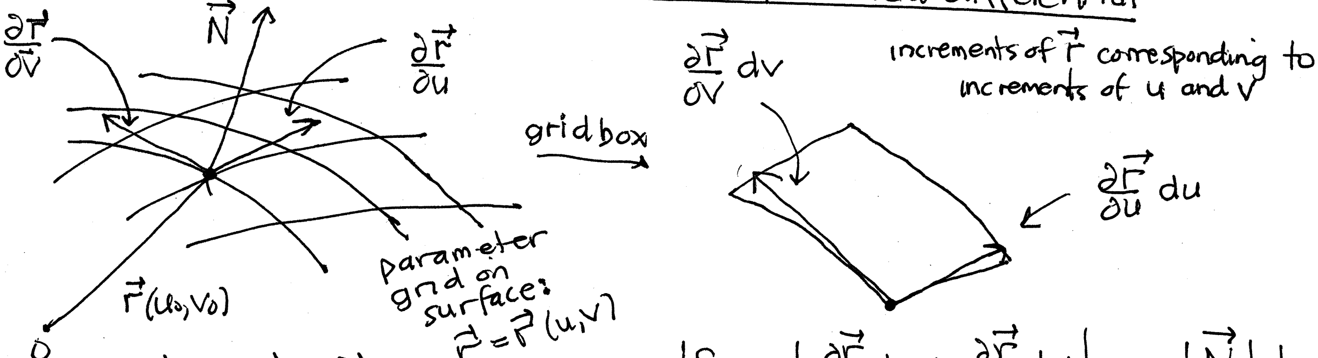
surface integral : normal vector and surface area differential



$$\vec N = \frac{\partial \vec r}{\partial u} \times \frac{\partial \vec r}{\partial v}$$

$$\hat N = \frac{\vec N}{|\vec N|}$$

$$dS = \left| \frac{\partial \vec r}{\partial u} du \times \frac{\partial \vec r}{\partial v} dv \right| = \underbrace{|\vec N|}_{\sqrt{\vec N \cdot \vec N}} du\,dv$$

cross-product gives area of parallelogram they determine.

typically complicated to integrate, like arclength!

surface integrals integrate the normal component of a vector field against the differential of surface area.

$$\iint_{\Sigma} \underbrace{\vec F(\vec r(u,v)) \cdot \hat N(\vec r(u,v)) \underbrace{dS}_{|\vec N(\vec r(u,v))| du\,dv}}_{\vec N(\vec r(u,v))}$$

Σ surface

$$= \int_{v_1}^{v_2}\int_{u_1}^{u_2} \vec F(\vec r(u,v)) \cdot \vec N(\vec r(u,v))\, du\,dv$$

$$\left[\int_C \underbrace{\vec F(\vec r(t)) \cdot \hat T(\vec r(t)) \underbrace{ds}_{|\vec r\,'(t)|\,dt}}_{\vec r\,'(t)} = \int_{t_1}^{t_2} \vec F(\vec r(t)) \cdot \vec r\,'(t)\, dt \right]$$

just like for the line integral case

square root goes away for vector integrals

easier than integrating scalars

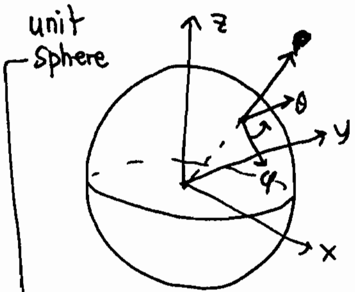
Surface integrals

$$\iint_{\mathcal{S}} \vec{F}\cdot d\vec{S} = \iint_{\mathcal{S}} \vec{F}\cdot\hat{n}\,\underbrace{dS}_{\|\vec{N}\|dA} = \iint_{uv\text{region}} \vec{F}(\vec{r}(u,v))\cdot\vec{N}(u,v)\,dA$$

$\vec{N}$

$$\left.\begin{aligned}\vec{r}&=\vec{r}(u,v)\\ \vec{N}&=\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\\ dA&=\text{"}du\,dv\text{"}\end{aligned}\right\}$$

unit sphere



$x^2+y^2+z^2=1$

outer normal

$z=\pm\sqrt{1-x^2-y^2}$

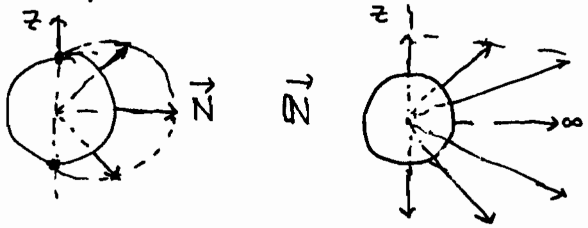
natural parametrization (ϕ,θ)

$\vec{r}=[\sin\phi\cos\theta,\ \sin\varphi\sin\theta,\ \cos\varphi]$

$\frac{\partial\vec{r}}{\partial\phi}=[-\cos\varphi\cos\theta,\ \cos\varphi\sin\theta,\ -\sin\varphi]$

$\frac{\partial\vec{r}}{\partial\theta}=[-\sin\varphi\sin\theta,\ \sin\varphi\cos\theta,\ 0]$

$$\vec{N}=\begin{vmatrix} i & j & k\\ -C_\varphi C_\theta & C_\varphi S_\theta & -S_\varphi\\ -S_\varphi S_\theta & S_\varphi C_\theta & 0\end{vmatrix}=\dots=\sin\varphi\,\underbrace{\vec{r}}_{=\hat{r}}$$



function graph approach (x,y)

upper hemisphere

$\vec{r}=[x,y,\sqrt{1-x^2-y^2}]$

$\frac{\partial\vec{r}}{\partial x}=[1,0,\frac{-x}{\sqrt{1-x^2-y^2}}]$

$\frac{\partial\vec{r}}{\partial y}=[0,1,\frac{-y}{\sqrt{1-x^2-y^2}}]$

$$\vec{N}=\frac{\partial\vec{r}}{\partial x}\times\frac{\partial\vec{r}}{\partial y}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\ 1&0&\frac{-x}{\sqrt{1-x^2-y^2}}\\ 0&1&\frac{-y}{\sqrt{1-x^2-y^2}}\end{vmatrix}$$

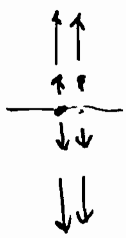
$$=\left[+\frac{x}{\sqrt{1-x^2-y^2}},\frac{+y}{\sqrt{1-x^2-y^2}},1\right]$$

$$=\frac{[x,y,\sqrt{1-x^2-y^2}]}{\sqrt{1-x^2-y^2}}\Big\}\leftarrow \vec{r}=\hat{r}$$

lower hemisphere:

$$\vec{N}=\frac{[x,y,-\sqrt{1-x^2-y^2}]}{\sqrt{1-x^2-y^2}}$$

$\vec{F}=[0,0,z]$



$\vec{F}(r(\varphi,\theta))=[0,0,\cos\varphi]$

$\vec{F}(r(\varphi,\theta))\cdot\vec{N}(\varphi,\theta)=\cos\varphi(\sin\varphi)(\cos\varphi)$

$$\iint_{\mathcal{S}}\vec{F}\cdot d\vec{S}=\int_0^{2\pi}\int_0^{\pi}\cos^2\varphi\sin\varphi\,d\varphi$$

$$=2\pi\left(-\frac{\cos^3\varphi}{3}\right)\Big|_0^{\pi}=\frac{4\pi}{3}$$

upper:

$\vec{F}(\vec{r}(x,y))=[0,0,\sqrt{1-x^2-y^2}]$

$\vec{F}(\vec{r}(x,y))\cdot\vec{N}(x,y)=\sqrt{1-x^2-y^2}$

$$\iint_{\mathcal{S}}\vec{F}\cdot d\vec{S}=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\sqrt{1-x^2-y^2}\,dy\,dx$$

$$=\int_0^{2\pi}\int_0^1\underbrace{\sqrt{1-r^2}}_{u}\,\underbrace{r\,dr}_{-\frac{1}{2}du}\,d\theta$$

$$(2\pi)\left(-\frac{1}{2}\,\frac{2}{3}(1-r^2)^{3/2}\Big|_0^1\right)=\frac{2\pi}{3}$$

lower $\vec{F}(\vec{r}(x,y))=[0,0,-\sqrt{1-x^2-y^2}]$

$\vec{F}(\vec{r}(x,y))\cdot\vec{N}(x,y)=\sqrt{1-x^2-y^2}$

$\rightarrow$ same result, so double $\rightarrow$ $\frac{4\pi}{3}$ ✓

scalar integral over upper hemisphere:

$$\iint_{\mathcal{S}} \underbrace{z}_{\cos\varphi=\sqrt{1-x^2-y^2}}\,dS \qquad dS=\|\vec{N}\|dA=\sin\varphi\,d\varphi\,d\theta=\frac{1}{\sqrt{1-x^2-y^2}}\,dy\,dx$$

$$=\int_0^{2\pi}\int_0^{\pi/2}\cos\varphi\sin\varphi\,d\varphi\,d\theta \qquad =\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\sqrt{1-x^2-y^2}\,\frac{1}{\sqrt{1-x^2-y^2}}\,dy\,dx$$

$$=2\pi\,\frac{\sin^2\varphi}{2}\Big|_0^{\pi/2}=\pi \qquad =\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}1\,dy\,dx=\pi(1)^2$$

$$=\int_0^{2\pi}\int_0^1 r\,dr\,d\theta$$

$$=2\pi\,\frac{r^2}{2}\Big|_0^1=\pi$$

$$\iint_{\mathcal{S}}dS=\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\sin\varphi\,d\varphi\,d\theta$$

$$=2\pi(-\cos\varphi)\Big|_0^{\pi/2}=2\pi$$

$$\iint_{\mathcal{S}}dS=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\frac{1}{\sqrt{1-x^2-y^2}}\,dy\,dx$$

$$=\int_0^{2\pi}\int_0^1\frac{r\,dr\,d\theta}{\sqrt{1-r^2}}=2\pi(-1)(1-r^2)^{1/2}\Big|_0^1$$

$$=2\pi$$

$$\bar{z}=\frac{\iint_{\mathcal{S}}z\,dS}{\iint_{\mathcal{S}}dS}=\frac{\pi}{2\pi}=\frac{1}{2}$$

center of gravity at $[0,0,\frac{1}{2}]$

Note $\operatorname{div}\vec{F}=\frac{\partial z}{\partial z}=1$

$$\iiint_{B}\operatorname{div}\vec{F}\,dV=V=\frac{4\pi}{3}(1)^3=\frac{4\pi}{3}$$

B ← ball inside sphere