

Grad, Curl, Div stuff (1) (vector language rules!)

notation: gradient $f \rightarrow \vec{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \equiv \underbrace{\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle}_\nabla f$

"del" is a vector derivative operator on scalars

$$\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \quad \text{applied to a function } f(x, y, z)$$

on its right like in 1-d:

$\frac{d}{dx} (\dots)$ acts on $f(x)$ to its right

combining $\vec{\nabla}$ with the two vector operations "dot" and "cross" yields two vector operators on vector fields.

$$\text{grad } f = \vec{\nabla} f$$

$$\text{div } \vec{F} = \vec{\nabla} \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle F_1, F_2, F_3 \rangle = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\text{curl } \vec{F} = \vec{\nabla} \times \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle F_1, F_2, F_3 \rangle$$

$$= \left\langle \frac{\partial F_3 - \partial F_2}{\partial y - \partial z}, \frac{\partial F_1 - \partial F_3}{\partial z - \partial x}, \frac{\partial F_2 - \partial F_1}{\partial x - \partial y} \right\rangle \quad \begin{matrix} \text{(clock} \\ \text{permutations)} \end{matrix}$$

all act from left to right on expressions (scalar/vector)

We can only combine these in succession in the combinations

$$\text{div grad } f = \vec{\nabla} \cdot \vec{\nabla} f = \vec{\nabla}^2 f = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \quad \text{"Laplacian"}$$

$$\text{curl grad } \vec{F} = \vec{\nabla} \times \vec{\nabla} f = 0 \quad \left. \begin{matrix} \text{identically zero for differentiable} \\ \text{scalar, vector fields} \end{matrix} \right\}$$

$$\text{div curl } \vec{F} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0$$

next page!

If $\vec{F} = \vec{\nabla} f$ is a conservative vector field then

$$\vec{\nabla} \times \vec{F} = \left\langle \underbrace{\frac{\partial(\partial f)}{\partial y(\partial z)} - \frac{\partial(\partial f)}{\partial z(\partial y)}}_{=0}, \underbrace{\frac{\partial(\partial f)}{\partial z(\partial x)} - \frac{\partial(\partial f)}{\partial x(\partial z)}}_{=0}, \underbrace{\frac{\partial(\partial f)}{\partial x(\partial y)} - \frac{\partial(\partial f)}{\partial y(\partial x)}}_{=0} \right\rangle$$

conversely if this condition is satisfied in a simply connected open set of space, a scalar potential exists.

scalar potentials easier to deal with than vector fields! energy versus force — physically important

ELECTRIC
GRAVITY
FIELDS
etc

Schroedinger's equation underlying all of CHEMISTRY!

Grad, Curl, DIV stuff (2)

If $\vec{B} = \text{curl } \vec{A} = \vec{\nabla} \times \vec{A}$, then

$$\text{div } \vec{B} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = \frac{\partial}{\partial x} \left(\frac{\partial A_3 - \partial A_2}{\partial y} \right) + \frac{\partial}{\partial y} \left(\frac{\partial A_1 - \partial A_3}{\partial z} \right) + \frac{\partial}{\partial z} \left(\frac{\partial A_2 - \partial A_1}{\partial x} \right) = 0$$

The magnetic field has zero divergence, can be represented as the curl of a "vector potential!"

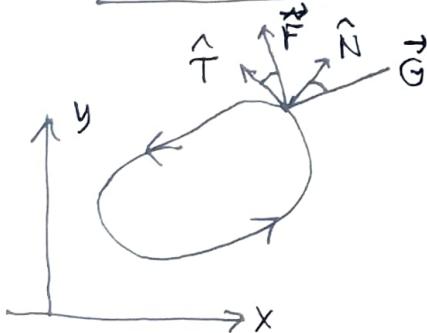
Green's Theorem

If $\vec{F}(x, y, z) = \langle F_1(x, y), F_2(x, y), 0 \rangle$ it is essentially a 2-d vector field!

$$\vec{\nabla} \times \vec{F} = \left\langle \frac{\partial F_3 - \partial F_2}{\partial y}, \frac{\partial F_1 - \partial F_3}{\partial z}, \frac{\partial F_2 - \partial F_1}{\partial x} \right\rangle = \hat{k} \left(\frac{\partial F_2 - \partial F_1}{\partial x} \right)$$

$$\hat{k} \cdot \vec{\nabla} \times \vec{F} = \hat{k} \cdot \text{curl } \vec{F} = \frac{\partial F_2 - \partial F_1}{\partial x} \quad \text{integrand in Green's Thm.}$$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \hat{k} \cdot \text{curl } \vec{F} \, dA = \oint_C \vec{F} \cdot \hat{T} \, ds \quad \begin{matrix} \uparrow \nearrow \\ \text{(tangential component)} \end{matrix}$$



see handout discussion (Green's Thm interpretation)

$$T, F_1, F_2 \xrightarrow[\text{90}^\circ \text{ clockwise}]{} \hat{T}, \hat{N}, G_1, G_2$$

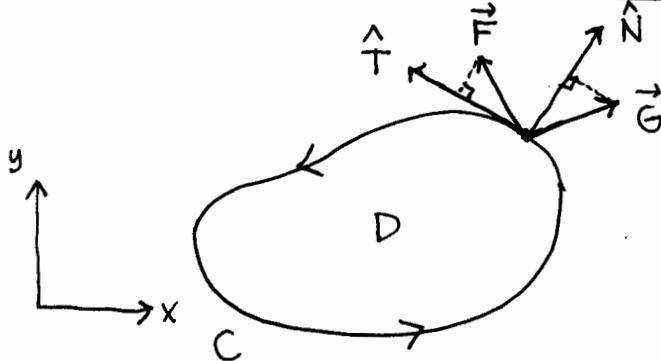
$$\vec{F} \cdot \hat{T} = \vec{G} \cdot \hat{N}$$

$$\text{curl } \vec{F} = \frac{\partial}{\partial x} (G_1) - \frac{\partial}{\partial y} (G_2) = \frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} = \text{div } \vec{G}$$

$$\oint_C \vec{G} \cdot \hat{N} \, ds = \iint_R \text{div } \vec{G} \, dA \quad \begin{matrix} \uparrow \\ \text{(normal component)} \end{matrix}$$

see handout discussion (div/curl interpretation)

geometrical interpretation of Green's Theorem: Gauss and Stokes



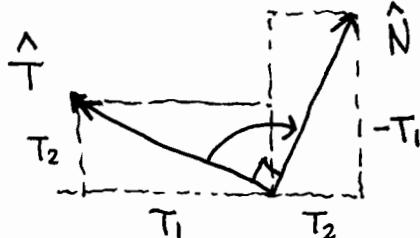
■ Green's Theorem:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} dA$$

$$\oint_C \vec{F} \cdot \hat{T} ds \quad \text{geometrical interpretation of LHS}$$

closed curve with counterclockwise orientation (direction)
enclosing a region D of the plane

let \hat{N} be the outer unit normal (points away from D)
obtained from the unit tangent \hat{T} by a 90° clockwise rotation:



$$\hat{T} = \langle T_1, T_2 \rangle \rightarrow \hat{N} = \langle N_1, N_2 \rangle = \langle T_2, -T_1 \rangle$$

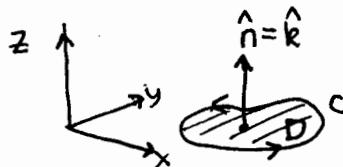
any vector field in the plane can be rotated pointwise by 90°

$$\vec{F} = \langle F_1, F_2 \rangle \rightarrow \vec{G} = \langle G_1, G_2 \rangle = \langle F_2, -F_1 \rangle$$

$$F_{\parallel} = \vec{F} \cdot \hat{T} = \vec{G} \cdot \hat{N} = G_{\perp}$$

(since rotation does not change the dot product between 2 vectors:
 $\hat{N} \cdot \vec{G} = N_1 G_1 + N_2 G_2 = T_2 F_2 + (-T_1) (-F_1) = T_1 F_1 + T_2 F_2 = \vec{F} \cdot \vec{F}$)

so tangential component of \vec{F} equals the (outer) normal component of \vec{G}
(scalars; can be positive, negative or zero)



D is a surface in space.
its upward normal \hat{n} obeys the
right-hand rule (RHR): curl fingers
in direction of C, thumb points upwards

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \hat{k} \cdot (\nabla \times \vec{F}) = \hat{k} \cdot \text{curl } \vec{F} \\ = \hat{n} \cdot \text{curl } \vec{F}$$

■ Stoke's theorem version:

$$\oint_C \vec{F} \cdot \hat{T} ds = \iint_D (\text{curl } \vec{F}) \cdot \hat{n} dA$$

integral of tangential component of vector field around
closed loop equals the integral of the RHR normal
component of its curl over the surface enclosed by the
curve

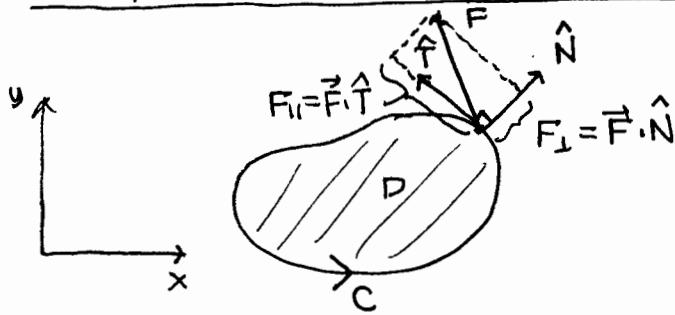
$$\frac{\partial G_1}{\partial x} - \frac{\partial (-G_2)}{\partial y} = \frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} = \nabla \cdot \vec{G} = \text{div } \vec{G}$$

■ Gauss's law version

$$\oint_C \vec{G} \cdot \hat{N} ds = \iint_D \text{div } \vec{G} dA$$

integral of the outer normal component of a vector field
around a closed loop equals the integral of its
divergence over the interior of the loop

interpretation of divergence and curl (2d)



C: counter clockwise loop with interior D

T-hat: unit counterclockwise tangent

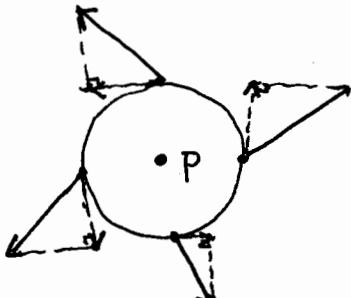
N-hat: unit outer normal

$F_{\parallel} = \vec{F} \cdot \hat{T}$ tangential component along curve

$F_{\perp} = \vec{F} \cdot \hat{N}$ normal component along curve

Green-Stokes

$$\oint_C \vec{F} \cdot \hat{T} ds = \text{circulation of } \vec{F} \text{ (counterclockwise) around } C = \iint_D \text{curl}(\vec{F})_z dA$$



To interpret the value of $\text{curl}(\vec{F})_z$ at a point P, shrink a small loop (circle or rectangle) down around P until the value of $\text{curl}(\vec{F})_z$ across the loop is almost constant (within some tolerance).

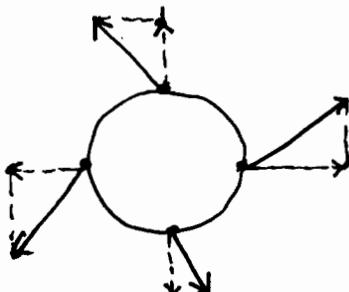
only F_{\parallel} contributes to the circulation around a loop

$$\approx \text{curl}(\vec{F})_z|_P \cdot \text{area}(D)$$

if nonzero there must always be a net circulation around the loop proportional to the area of the loop, in the counterclockwise sense if positive, clockwise if negative.

Green-Gauss

$$\oint_C \vec{F} \cdot \hat{N} ds = \text{net flux of } \vec{F} \text{ out of loop } C = \iint_D \text{div}(\vec{F}) dA$$



Again shrink a loop down around P until the value of $\text{div}(\vec{F})$ is almost constant across the loop

only F_{\perp} contributes to the flux in or out of the loop

$$\approx \text{div}(\vec{F})|_P \cdot \text{area}(D)$$

if nonzero there must always be a net flux in (if negative) or out (if positive) of the loop

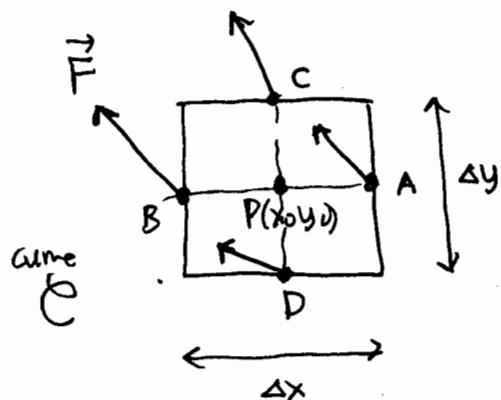
interpretation of divergence and curl (2d) continued

WARNING! PROCEED AT YOUR OWN RISK. OPTIONAL MATERIAL

When one first sees these things in physics for the first time a student has no clue how to evaluate a line or flux integral except if the tangential or normal component is a constant and the curve is a circle so calculus is unnecessary (!).

Starting from this lack of knowledge one can then derive component formulas for the divergence and curl from a limiting ratio.

Here we examine $\text{curl}(\vec{F})_z$ and the circulation, using a rectangular curve C of width Δx and height Δy . We approximate \vec{F} by pretending it is constant on each side of the box using its midpoint value for the approximate constant value & evaluate those 4 values using Taylor series centered at P .



counterclockwise tangential components:

- side A : $F_y(A)$ length Δy
- side B : $-F_y(B)$ length Δy
- side C : $-F_x(C)$ length Δx
- side D : $F_x(D)$ length Δx

(physics vector notation:
 $\vec{F} = \langle F_x, F_y, F_z \rangle$)

$$\oint_C \vec{F} \cdot \hat{T} ds \approx \underbrace{\left[F_y(x_0, y_0) + \frac{\partial F_y}{\partial x}(x_0, y_0) \left(\frac{\Delta x}{2} \right) \right] \Delta y}_{\text{value at midpt A}} - \underbrace{\left[F_x(x_0, y_0) + \frac{\partial F_x}{\partial y}(x_0, y_0) \left(\frac{\Delta y}{2} \right) \right] \Delta x}_{\text{value at midpt C}} \\ - \underbrace{\left[F_y(x_0, y_0) + \frac{\partial F_y}{\partial x}(x_0, y_0) \left(-\frac{\Delta x}{2} \right) \right] \Delta y}_{\text{value at midpt B}} + \underbrace{\left[F_x(x_0, y_0) + \frac{\partial F_x}{\partial y}(x_0, y_0) \left(-\frac{\Delta y}{2} \right) \right] \Delta x}_{\text{value at midpt D}}$$

$$\begin{aligned} \text{sum: } &= 0 + \frac{\partial F_y}{\partial x}(x_0, y_0) \Delta x \Delta y + 0 - \frac{\partial F_x}{\partial y}(x_0, y_0) \Delta x \Delta y \\ &= \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)(x_0, y_0) \Delta x \Delta y \end{aligned}$$

In the limit C shrinks to P we obtain

$$\text{curl}(\vec{F})_z = \lim_{C \rightarrow P} \frac{\oint_C \vec{F} \cdot \hat{T} ds}{\text{area}(C)} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}$$

a similar calculation for the outward flux integral produces

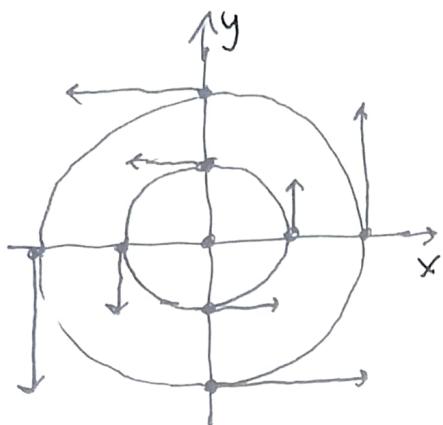
$$\text{div}(\vec{F}) = \lim_{C \rightarrow P} \frac{\oint_C \vec{F} \cdot \hat{N} ds}{\text{area}(C)} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}$$

Grad, curl, Div stuff (8)

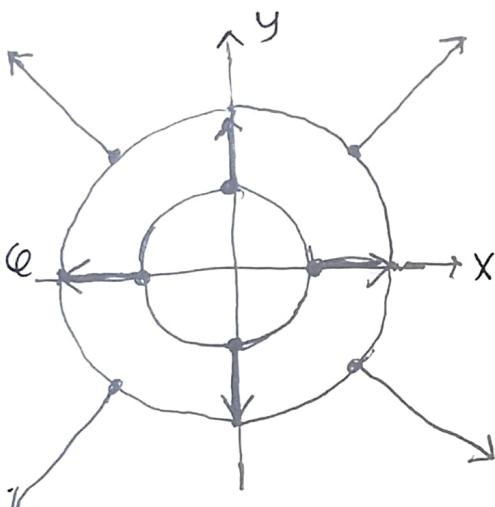
page 6!

examples of purely curling / purely diverging vector fields

pure
curl
field
(zero
(divergence)



pure
divergence
field
(zero
(curl)



$$\vec{F} = \langle -y, x \rangle \quad \text{tangential to circles:}$$

$$= \hat{r} \times \vec{r} \quad \vec{F} \cdot \langle r, 0 \rangle = 0$$

$$\hat{r} \cdot \text{curl } \vec{F} = \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y) = 2 \text{ constant}$$

$$\text{div } \vec{F} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x) = 0$$

"circulating pattern"

moving around origin
counterclockwise

(flip sign,
clockwise)

$$\vec{F} = \langle x, y \rangle = \vec{r}$$

$$\hat{r} \cdot \text{curl } \vec{F} = \frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x) = 0$$

$$\text{div } \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 2 \text{ constant}$$

"divergent" spray of vectors

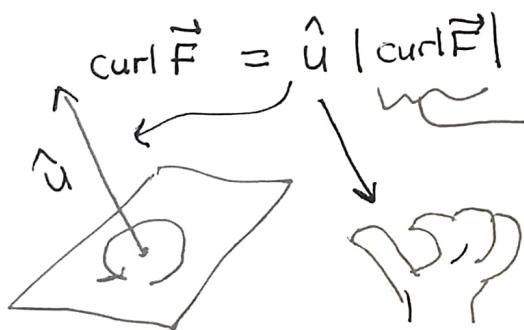
coming out of origin

"source"

(flip sign, goes into origin)

"sink"

3d
curl



magnitude measures circulation in
plane orthogonal to u-hat

right hand rule
gives direction
of circulation about direction u-hat

Grad, Curl, Div stuff (9)

page 7!

inverse square force field: $f = \frac{k}{|\vec{r}|} \rightarrow \vec{F} = \vec{\nabla}f = -\frac{k\hat{r}}{|\vec{r}|^2} = -\frac{k\vec{r}}{|\vec{r}|^3}$

$$= -\frac{k(x_1 y_1 z)}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\frac{\partial F_1}{\partial x} = \frac{\partial}{\partial x} \left(\frac{-kx}{(x^2 + y^2 + z^2)^{3/2}} \right) = -k \left[\frac{(x^2 + y^2 + z^2)^{3/2} (1) - x \frac{3}{2} (x^2 + y^2 + z^2)^{1/2} (2x)}{(x^2 + y^2 + z^2)^3} \right] \text{quotient rule}$$

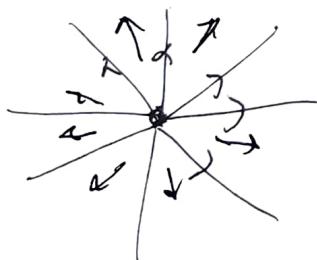
$$= -k \frac{(x^2 + y^2 + z^2)^{1/2} ((x^2 + y^2 + z^2) - 3x^2)}{(x^2 + y^2 + z^2)^3} = -\frac{k(x^2 + y^2 + z^2 - 3x^2)}{(x^2 + y^2 + z^2)^{5/2}}$$

$$\frac{\partial F_2}{\partial y} = \dots = -k \frac{(x^2 + y^2 + z^2 - 3y^2)}{(x^2 + y^2 + z^2)^{5/2}}$$

$$\frac{\partial F_3}{\partial z} = \dots = -k \frac{(x^2 + y^2 + z^2 - 3z^2)}{(x^2 + y^2 + z^2)^{5/2}}$$

$$\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = -k \left(\frac{3(x^2 + y^2 + z^2) - 3(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{5/2}} \right) = 0 \text{ if } \vec{r} \neq 0!$$

BUT $k < 0$ outward radial force field (like positive charge)



diverging out from origin!

∞ -divergence at origin \rightarrow location of point charge

we need surface integrals to clarify this apparent contradiction.

$\operatorname{div} \vec{E} = 4\pi\rho$ (chargedensity, infinite at origin!)

electric field lines start or end at charge ("source")

$\operatorname{div} \vec{B} = 0$ magnetic field lines never begin or end since magnetic charge does not exist!