inverse power radial force fields

$$\vec{r} = \langle x, y, z\rangle \qquad |\vec{r}| = (x^2+y^2+z^2)^{1/2} = \rho$$

$$\vec{F} = \frac{\vec{r}}{|\vec{r}|^p} = \frac{\langle x,y,z\rangle}{(x^2+y^2+z^2)^{p/2}} = \frac{\hat{r}}{|\vec{r}|^{p-1}} = \frac{\hat{r}}{\rho^{p-1}}$$

$p=3$ for inverse square force field.

$p>1$ for decreasing magnitude with distance.

$$\frac{\partial F_1}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{p/2}}\right)$$

$$= \frac{(x^2+y^2+z^2)^{p/2}\cdot 1 - x\,\frac{p}{2}(x^2+y^2+z^2)^{\frac{p}{2}-1}(2x)}{(x^2+y^2+z^2)^p}$$

$$= \frac{(x^2+y^2+z^2)^{\frac{p}{2}-1}(\overbrace{x^2+y^2+z^2}^{\rho^2} - p x^2)}{(x^2+y^2+z^2)^p}$$

$$= \frac{\rho^{p-2}}{\rho^{2p}}(\rho^2 - p x^2) = \frac{\rho^2 - p x^2}{\rho^{p+2}}$$

$$\frac{\partial F_2}{\partial y} = \frac{\rho^2 - p y^2}{\rho^{p+2}}$$

$$\frac{\partial F_3}{\partial z} = \frac{\rho^2 - p z^2}{\rho^{p+2}}$$

$$\operatorname{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{3\rho^2 - p\overbrace{(x^2+y^2+z^2)}^{\rho^2}}{\rho^{p+2}} = \frac{3-p}{\rho^p} = \frac{3-p}{|\vec{r}|^p}$$

only the inverse square force has zero divergence*

NOTE curl of any radial force field is zero.

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}\left(y(x^2+y^2+z^2)^{-p/2}\right) = y\left(-\frac{p}{2}\right)(x^2+y^2+z^2)^{-p/2-1}(2x)$$

$$= \frac{-pxy}{(x^2+y^2+z^2)^{p/2+1}}$$

$$\frac{\partial F_1}{\partial y} = \ldots = \frac{-pyx}{(x^2+y^2+z^2)^{p/2+1}}$$

$$\underbrace{\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}}_{(\operatorname{curl}\vec{F})_3} = 0$$ and cyclic permutations for remaining 2 components of curl.

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* on a sphere $\rho = \rho_0$, surface integral: $\iint_\Sigma \vec{F}\cdot\hat{N}\,dS = \int_0^{2\pi}\int_0^{\pi} \underbrace{\frac{1}{\rho^{p-1}}}_{\vec{F}\cdot\hat{r}} \underbrace{\rho^2 \sin\phi\, d\phi\, d\theta}_{dS} = 4\pi\frac{\rho^2}{\rho^{p-1}} = 4\pi\rho^{3-p}$

for $p>3$ this is a decreasing function of ρ so the net flux out of a sphere decreases, leading to a net inflow of flux between two such spheres, corresponding to the negative divergence. <0

$$\underset{\rho_1\to\rho_2}{\iiint} \operatorname{div}\vec{F}\,dV = \underbrace{\int_0^{2\pi}\int_0^{\pi}}_{\text{unit sphere}}\int_{\rho_1}^{\rho_2} \frac{3-p}{\rho^p}\underbrace{\rho^2\sin\phi\,d\phi\,d\theta}_{dV} = 4\pi\int_{\rho_1}^{\rho_2}(3-p)\rho^{2-p}\,d\rho = 4\pi(3-p)\left.\frac{\rho^{2-p+1}}{2-p+1}\right|_{\rho_1}^{\rho_2} = 4\pi\left(\frac{1}{\rho_1^{3-p}} - \frac{1}{\rho_2^{3-p}}\right)$$