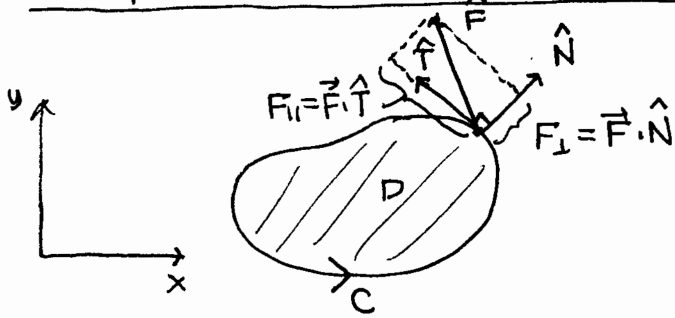
interpretation of divergence and curl (2d)



C: counterclockwise loop with interior D

$\hat{T}$: unit counterclockwise tangent

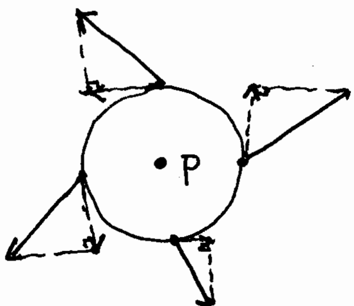
$\hat{N}$: unit outer normal

$F_{\parallel} = \vec{F} \cdot \hat{T}$ tangential component along curve

$F_{\perp} = \vec{F} \cdot \hat{N}$ normal component along curve

Green-Stokes

$$\oint_C \vec{F} \cdot \hat{T}\, ds = \text{circulation of } \vec{F} \text{ (counterclockwise) around } C = \iint_D \operatorname{curl}(\vec{F})_z \, dA$$



To interpret the value of $\operatorname{curl}(F)_z$ at a point P, shrink a small loop (circle or rectangle) down around P until the value of $\operatorname{curl}(\vec{F})_z$ across the loop is almost constant (within some tolerance)

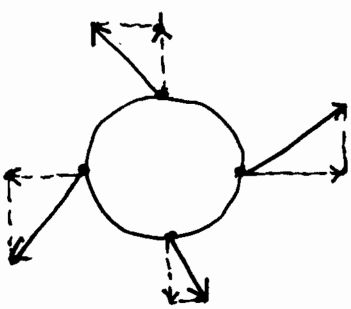
$$\approx \underbrace{\operatorname{curl}(\vec{F})_z \Big|_P}_{} \cdot \operatorname{area}(D)$$

only $F_{\parallel}$ contributes to the circulation around a loop

if nonzero there must always be a net circulation around the loop proportional to the area of the loop, in the counterclockwise sense if positive, clockwise if negative.

Green-Gauss

$$\oint_C \vec{F} \cdot \hat{N}\, ds = \text{net flux of } \vec{F} \text{ out of loop } C = \iint_D \operatorname{div}(\vec{F})\, dA$$



Again shrink a loop down around P until the value of $\operatorname{div}(\vec{F})$ is almost constant across the loop

$$\approx \underbrace{\operatorname{div}(F) \Big|_P}_{} \cdot \operatorname{area}(D)$$

only $F_{\perp}$ contributes to the flux in or out of the loop

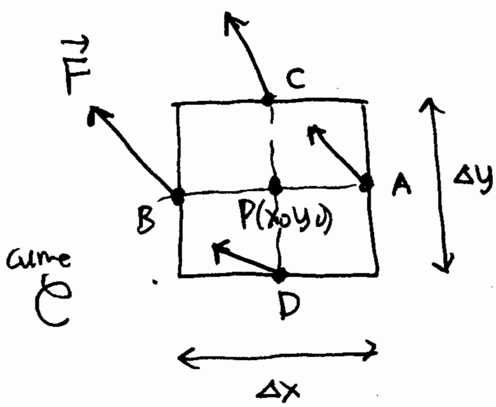
if nonzero there must always be a net flux in (if negative) or out (if positive) of the loop interpretation of divergence and curl (2d) continued

WARNING! PROCEED AT YOUR OWN RISK. OPTIONAL MATERIAL

When one first sees these things in physics for the first time a student has no clue how to evaluate a line or flux integral except if the tangential or normal component is a constant and the curve is a circle so calculus is unnecessary (!).

Starting from this lack of knowledge one can then derive component formulas for the divergence and curl from a limiting ratio.

Here we examine $\text{curl}(\vec{F})_z$ and the circulation, using a rectangular curve C of width Δx and height Δy. We approximate $\vec{F}$ by pretending it is constant on each side of the box using its midpoint value for the approximate constant value & evaluate those 4 values using Taylor series centered at P.



counterclockwise tangential components:

side A :	$F_y(A)$	length Δy
side B :	$-F_y(B)$	length Δy
side C :	$-F_x(C)$	length Δx
side D :	$F_x(D)$	length Δx

(physics vector notation: $\vec{F} = \langle F_x, F_y, F_z \rangle$)

$$\oint_{\text{curve}} \vec{F}\cdot\hat{T}\,ds \approx \underbrace{\left[F_y(x_0y_0) + \frac{\partial F_y}{\partial x}(x_0y_0)\left(\frac{\Delta x}{2}\right)\right]}_{\text{value at midpt A}}\Delta y \quad - \underbrace{\left[F_x(x_0y_0) + \frac{\partial F_x}{\partial y}(x_0y_0)\left(\frac{\Delta y}{2}\right)\right]}_{\text{value at midpt C}}\Delta x$$

$$- \underbrace{\left[F_y(x_0y_0) + \frac{\partial F_y}{\partial x}(x_0y_0)\left(\frac{-\Delta x}{2}\right)\right]}_{\text{value at midpt B}}\Delta y \quad + \underbrace{\left[F_x(x_0y_0) + \frac{\partial F_x}{\partial y}(x_0y_0)\left(-\frac{\Delta y}{2}\right)\right]}_{\text{value at midpt D}}\Delta x$$

$$\text{sum:} \quad = \quad 0 \quad + \frac{\partial F_y}{\partial x}(x_0y_0)\,\Delta x\Delta y \quad + \quad 0 \quad - \frac{\partial F_x}{\partial y}(x_0y_0)\,\Delta x \Delta y$$

$$= \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)(x_0y_0)\,\Delta x\Delta y$$

In the limit $\mathcal{C}$ shrinks to P we obtain

$$\text{curl}(\vec{F})_z = \lim_{\mathcal{C}\to P}\frac{\oint_{\mathcal{C}} \vec{F}\cdot\hat{T}\,ds}{\text{area}(\mathcal{C})} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}$$

a similar calculation for the outward flux integral produces

$$\text{div}(\vec{F}) = \lim_{\mathcal{C}\to P}\frac{\oint \vec{F}\cdot\hat{N}\,ds}{\text{area}(\mathcal{C})} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}$$