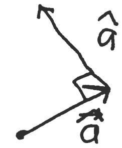
Geometry of Curves

preliminaries:

chain rule: $\dfrac{dQ}{ds} = \dfrac{dQ/dt}{ds/dt} = \dfrac{Q'}{s'}$

length direction decomposition: $\vec{a} \nearrow |\vec{a}|$, $\searrow \hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$ $\rightarrow \vec{a} = |\vec{a}|\,\hat{a}$

unit vector derivative: $\hat{a}\cdot\hat{a}' = 0$



$\vec{r} = \vec{r}(t) = \langle x(t), y(t), z(t)\rangle$

$\downarrow D$

$\vec{r}' \nearrow |\vec{r}'| = s'$, $\searrow \hat{T} = \dfrac{\vec{r}'}{|\vec{r}'|}$ $\rightarrow \vec{r}' = s'\hat{T} \xrightarrow{D} \vec{r}'' = s''\hat{T} + s'\hat{T}'$

$\vec{r}'\times\vec{r}'' = (s'\hat{T}) \times (s''\hat{T} + s'\hat{T}') = s'^2\, \hat{T}\times\hat{T}'$ ($\hat{T}\times\hat{T} \to 0$)

$|\vec{r}'\times\vec{r}''| = s'^2 \underbrace{|\hat{T}\times\hat{T}'|}_{|\hat{T}||\hat{T}'|\sin\frac{\pi}{2}} = s'^2 \underbrace{|\hat{T}'|}_{\kappa s'} = s'^3 \kappa$

$\hat{T}' \rightarrow \hat{N} = \dfrac{\hat{T}'}{|\hat{T}'|}$

$\left[\hat{B} = \dfrac{\vec{r}'\times\vec{r}''}{|\vec{r}'\times\vec{r}''|} = \hat{T}\times\hat{N}\right]$

$$\boxed{\kappa = \frac{|\vec{r}'\times\vec{r}''|}{s'^3} = \frac{|\vec{r}'\times\vec{r}''|}{|\vec{r}'|^3}}$$

$\dfrac{d\hat{T}}{ds} = \dfrac{\hat{T}'}{s'} \rightarrow \left|\dfrac{d\hat{T}}{ds}\right| = \boxed{\dfrac{|\hat{T}'|}{s'} = \kappa = \dfrac{|\hat{T}'|}{|\vec{r}'|}} = \dfrac{1}{\rho}$

HELIX EXAMPLE

$\vec{r} = \langle a\cos t, a\sin t, bt\rangle \quad a, b > 0$

$\vec{r}' = \langle -a\sin t, a\cos t, b\rangle \xrightarrow{\text{or}} \vec{r}'' = \langle -a\cos t, -a\sin t, 0\rangle$

$|\vec{r}'| = \sqrt{a^2s^2 + a^2c^2 + b^2} = \sqrt{a^2+b^2} = s'$

$\hat{T} = \dfrac{1}{\sqrt{a^2+b^2}}\langle -a\sin t, a\cos t, b\rangle$

$\hat{T}' = \dfrac{1}{\sqrt{a^2+b^2}}\langle -a\cos t, -a\sin t, 0\rangle$

$|\hat{T}'| = \dfrac{1}{\sqrt{a^2+b^2}}\sqrt{a^2c^2+a^2s^2} = \dfrac{a}{\sqrt{a^2+b^2}}$

$\kappa = \dfrac{|\hat{T}'|}{s'} = \dfrac{a/\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}} = \dfrac{a}{a^2+b^2} = \dfrac{1}{\rho}$

$\rho = \dfrac{a^2+b^2}{a} = a + \dfrac{b^2}{a} > a$

$\hat{N} = \hat{T}'/|\hat{T}'| = \dfrac{1/\sqrt{a^2+b^2}}{a/\sqrt{a^2+b^2}}\langle\cdots\rangle = \langle \underbrace{-\cos t, -\sin t}, \overset{\text{horizontal}}{0}\rangle$ points towards z-axis

$\hat{B} = \hat{T}\times\hat{N} = \dfrac{1}{\sqrt{a^2+b^2}}\begin{vmatrix} i & j & k \\ -as & ac & b \\ -c & -s & 0\end{vmatrix} = \dfrac{1}{\sqrt{a^2+b^2}}\langle 0 + bs, -bc - 0, a(s^2+c^2)\rangle$

$= \dfrac{1}{\sqrt{a^2+b^2}}\langle b\sin t, -b\cos t, a\rangle$

$\vec{r}'\times\vec{r}'' = \begin{vmatrix} i & j & k \\ -as & ac & b \\ -ac & -as & 0\end{vmatrix} = \langle 0 + abs, -abc - 0, a^2(s^2+c^2)\rangle$

$= \langle abs, -abc, a^2\rangle$

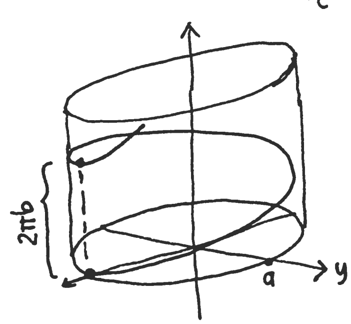
$= a\langle bs, -bc, a\rangle$

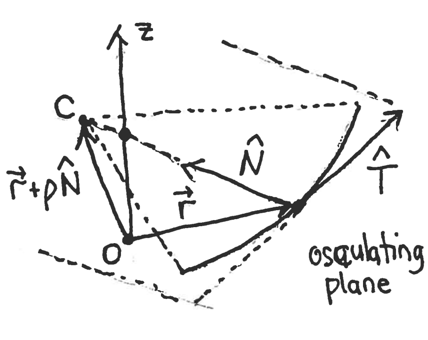
$|\vec{r}'\times\vec{r}''| = a\sqrt{b^2s^2 + b^2c^2 + a^2} = a\sqrt{a^2+b^2}$

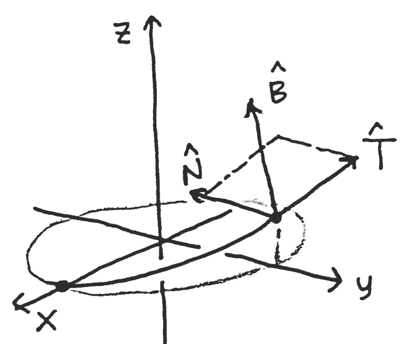
$\kappa = \dfrac{|\vec{r}'\times\vec{r}''|}{|\vec{r}'|^3} = \dfrac{a\sqrt{a^2+b^2}}{(a^2+b^2)^{3/2}} = \dfrac{a}{a^2+b^2}$

$\hat{B} = \dfrac{\vec{r}'\times\vec{r}''}{|\vec{r}'\times\vec{r}''|} = \dfrac{a}{a\sqrt{a^2+b^2}}\langle bs, -bc, a\rangle$

$= \dfrac{1}{\sqrt{a^2+b^2}}\langle b\sin t, -b\cos t, a\rangle$







$t = \frac{\pi}{2}$

space curvature and acceleration (1)

arclength $s = \int_{t_0}^{t} |\vec{r}\,'(t)|\,dt \leftrightarrow s' = \frac{ds}{dt} = |\vec{r}\,'(t)|$ (speed)

position: $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

velocity: $\vec{v}(t) = \vec{r}\,'(t)$ (tangent)
$\longrightarrow$ length: $|\vec{r}\,'(t)|$
$\longrightarrow$ direction: $\hat{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ (unit tangent)

$$\vec{r}\,'(t) = |\vec{r}\,'(t)|\,\hat{T}(t) = s'(t)\,\hat{T}(t)$$

acceleration $$\vec{a}(t) = \vec{r}\,''(t) = \frac{d}{dt}\left(s'(t)\hat{T}(t)\right) \underset{\text{prod rule}}{=} s''(t)\hat{T}(t) + s'(t)\underbrace{\hat{T}'(t)}_{\text{what's this?}}$$

$\hat{T}$ is a unit vector so it can only rotate $\Rightarrow$ $\hat{T}'$ is $\perp$ to $\hat{T}$

$$\frac{d}{dt}\left(\hat{T}\cdot\hat{T} = 1\right) \xrightarrow[\text{rule}]{\text{prod}} \hat{T}'\cdot\hat{T} + \hat{T}\cdot\hat{T}' = 0$$
$$2\hat{T}\cdot\hat{T}' = 0 \rightarrow \hat{T}\cdot\hat{T}' = 0$$

$\hat{T}'$ gives direction in which tip of $\hat{T}$ is rotating

so $$\hat{N}(t) = \frac{\hat{T}'(t)}{|\hat{T}'(t)|} = \frac{\frac{d}{dt}\left(\frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}\right)}{\left|\frac{d}{dt}\left(\frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}\right)\right|}$$ quotient rule + divide by length calculation = unit normal is direction in which $\hat{T}$ is rotating: $\hat{N}(t)\cdot\hat{T}(t) = 0$

But only arclength derivatives produce information about the path independent of how fast one moves along it.
CHAIN RULE allows us to compute such derivatives without having the curve parametrized by arclength s:
$$\frac{df}{ds} = \frac{df/dt}{ds/dt} = \frac{f'}{s'} = \frac{f'}{|\vec{r}\,'|}.$$

$$\frac{d\vec{r}}{ds} = \frac{\vec{r}\,'}{s'} = \frac{\vec{r}\,'}{|\vec{r}\,'|} = \hat{T} \quad \text{(as before)}$$

$$\frac{d^2\vec{r}}{ds^2} = \frac{d\hat{T}}{ds} = \underbrace{\left|\frac{d\hat{T}}{ds}\right|}_{\text{length}}\ \underbrace{\widehat{\frac{d\hat{T}}{ds}}}_{\text{direction}} = \kappa\hat{N}$$

$\kappa \geq 0$ curvature

$\hat{N}$ unit normal $= \frac{d\hat{T}/ds}{|d\hat{T}/ds|} = \frac{\hat{T}'}{|\hat{T}'|}$ (as before)

chain rule

$$= \frac{d\hat{T}/dt}{ds/dt} = \frac{\hat{T}'}{s'}$$

s' cancels out

$$\boxed{\hat{T}' = s'\kappa\hat{N}}$$ substitute above

radius of curvature $\kappa(t) = \frac{1}{\rho(t)}$

decomposition of acceleration

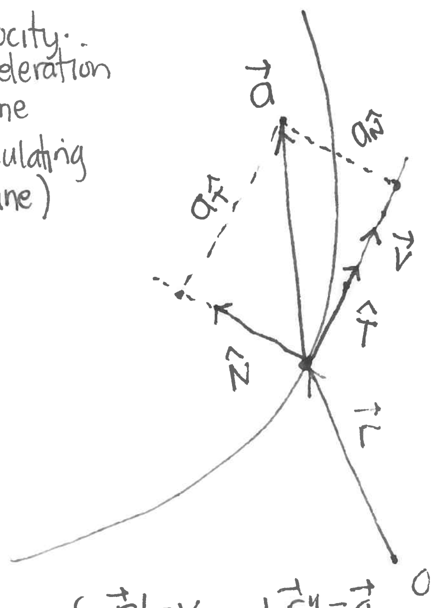
$$\underbrace{\vec{r}\,''(t)}_{\vec{a}(t)} = \underbrace{s''(t)}_{a_{\hat{T}}}\,\hat{T}(t) + \underbrace{s'(t)^2\kappa(t)}_{a_{\hat{N}}}\,\hat{N}(t)$$

$a_{\hat{T}}$ = linear acc. along direction of motion
"tangential acc"
$a_{\hat{T}} = \vec{a}\cdot\hat{T}$

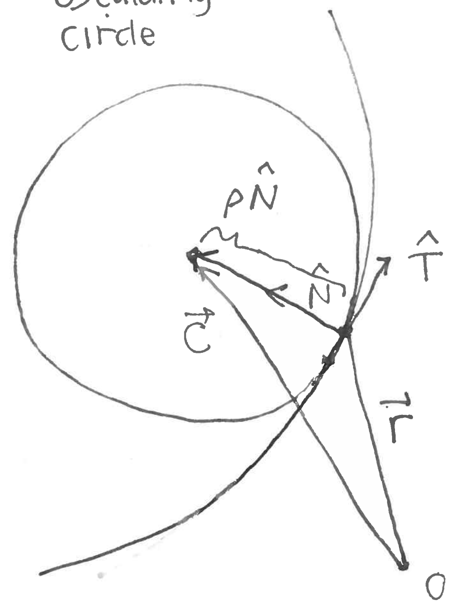
$a_{\hat{N}}$ = acc. perp to direction of motion $= \frac{s'(t)^2}{\rho(t)}$ $\left(\text{"}\frac{v^2}{r}\text{"}\right)$
"normal acceleration"
"centripetal acc."
$a_{\hat{N}} = \vec{a}\cdot\hat{N}$

space curvature and acceleration (2)

velocity-acceleration plane (osculating plane)



osculating circle



plane of $\vec{r}\,' = v$ and $\vec{r}\,'' = \vec{a}$
or of $\hat{T}$ and $\hat{N}$

normal: $\vec{r}\,' \times \vec{r}\,''$ or $\underbrace{\hat{T} \times \hat{N}}$

center is a distance ρ along $\hat{N}$ from the curve:

$$\vec{C} = \vec{r} + \rho\hat{N}$$

$\hat{B}$ = binormal = unit normal to vel-acc plane

$$= \frac{\vec{r}\,' \times \vec{r}\,''}{|\vec{r}\,' \times \vec{r}\,''|}$$

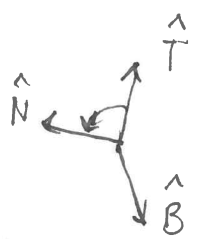
Note $|\hat{B}| = \underbrace{|\hat{T}|}_{1}\,\underbrace{|\hat{N}|}_{1}\,\underbrace{\sin\tfrac{\pi}{2}}_{1} = 1$
unit vector perp to $\hat{T}$ and $\hat{N}$

But $\underbrace{\vec{r}\,'}_{s'\hat{T}} \times \underbrace{\vec{r}\,''}_{s''\hat{T} + s'^2 kN} = s'\hat{T} \times (s''\hat{T} + s'^2 kN) = s's''\underbrace{\hat{T}\times\hat{T}}_{0} + s'^3 k \underbrace{\hat{T}\times N}_{\hat{B}}$

$$|\vec{r}\,' \times \vec{r}\,''| = s'^3 k \underbrace{|\hat{T}\times\hat{N}|}_{1} \rightarrow \boxed{k = \frac{|r' \times r''|}{s'^3} = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3}}$$

1-2-3 rule

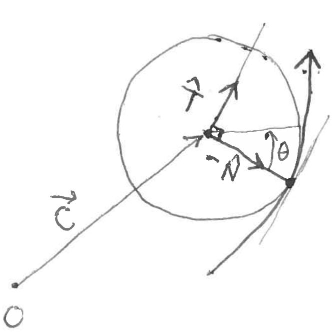
avoids quotient rule computation



right hand rule:

$$\hat{N} = \hat{B} \times \hat{T}$$

compute $\hat{N}$ by cross-product instead of $\hat{T}\,'$ derivative calculation too.



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The parametrization of the osculating circle is straightforward. Locate the vectors $(-\hat{N})$ and $\hat{T}$ at its center and use the correspondence $-\hat{N} \leftrightarrow \hat{\imath}$, $\hat{T} \leftrightarrow \hat{\jmath}$ with the unit vectors in the x-y plane:

$$\vec{r}_c(\theta) = \underbrace{\vec{C}}_{\text{new center}} + \underbrace{\rho\cos\theta(-\hat{N}) + \rho\sin\theta\,\hat{T}}, \quad 0 \le \theta \le 2\pi$$

$\sim \vec{r}(\theta) = \langle \rho\cos\theta, \rho\sin\theta \rangle = \rho\cos\theta\,\hat{\imath} + \rho\sin\theta\,\hat{\jmath}$ in xy plane at origin

space curvature and acceleration (3)

twisted cubic example (rescaled to make perfect square ↓)

$$\vec r = \langle 2t, t^2, t^3/3\rangle$$

$$\vec r\,' = \langle 2, 2t, t^2\rangle \qquad |\vec r\,'| = \sqrt{4+4t+t^4} = \sqrt{(t^2+2)^2} = t^2+2$$

$$\hat T = \frac{\vec r\,'}{|\vec r\,'|} = \frac{\langle 2,2t,t^2\rangle}{t^2+2}$$

$$\vec r\,'' = \langle 0,2,2t\rangle = 2\langle 0,1,t\rangle$$

$$\vec r\,'\times\vec r\,'' = 2\langle 2,2t,t^2\rangle \times \langle 0,1,t\rangle = 2\begin{vmatrix} i & j & k \\ 2 & 2t & t^2 \\ 0 & 1 & 1\end{vmatrix} = 2\langle 2t^2-t^2, 0-2t, 2-0\rangle = 2\langle t^2,-2t,2\rangle$$

$$|\vec r\,'\times\vec r\,'''| = 2\sqrt{t^4+4t^2+4} = 2(t^2+2) \quad \text{(as before)}$$

$$K = \frac{|r'\times r'''|}{|r'|^3} = \frac{2(t^2+2)}{(t^2+2)^3} = \frac{2}{(t^2+2)^2} \to \rho = \frac{(t^2+2)^2}{2}$$

$$\hat B = \frac{\vec r\,'\times\vec r\,''}{|r'\times r''|} = \frac{2\langle t^2,-2t,2\rangle}{2\ (t^2+2)} = \frac{\langle t^2,-2t,2\rangle}{t^2+2}$$

$$\hat N = \hat B\times\hat T = \frac{1}{(t^2+2)}\frac{1}{(t^2+2)}\begin{vmatrix} i & j & k \\ t^2 & -2t & 2 \\ 2 & 2t & t^2\end{vmatrix} = \frac{1}{(t^2+2)^2}\langle \underbrace{-2t^3-4t}_{-2t(t^2+2)}, \underbrace{4-t^4}_{(2-t^2)(2+t^2)}, \underbrace{2t^3+4t}_{2t(t^2+2)}\rangle$$

$$= \frac{\langle -2t, 2-t^2, 2t\rangle}{t^2+2}$$

or $\hat T = \dfrac{\langle 2,2t,t^2\rangle}{t^2+2}$

$$\hat T\,' = \frac{(t^2+2)\langle 0,2,2t\rangle - \langle 2,2t,t^2\rangle(2t)}{(t^2+2)^2}$$

$$= \frac{\langle 0, 2t^2+4, 2t^3+4t\rangle - \langle 4t, 4t^2, 2t^3\rangle}{(t^2+2)^2}$$

$$= \frac{\langle -4t, 4-2t^2, 4t\rangle}{(t^2+2)^2} = \frac{2}{(t^2+2)^2}\langle -2t, 2-t^2, 2t\rangle$$

so $\hat N = \hat T\,'/|\hat T\,'| = \dfrac{\langle -2t, 2-t^2, 2t\rangle}{\sqrt{4t^2+(4-4t^2+t^4)+4t^2}}$

$\sqrt{(t^2+2)^2} = t^2+2$

at $t=0$:

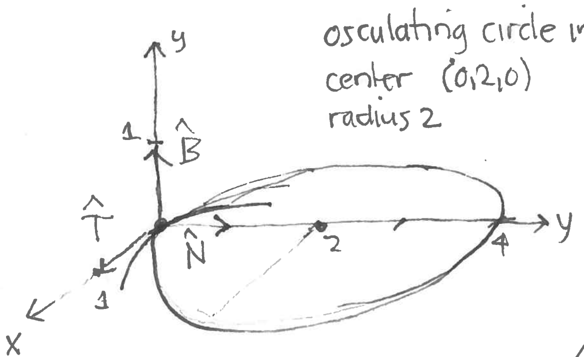
$\vec r(0) = \langle 0,0,0\rangle$

$\hat T(0) = \dfrac{\langle 2,0,0\rangle}{2} = \langle 1,0,0\rangle = \hat i$

$\hat N(0) = \dfrac{\langle 0,2,0\rangle}{2} = \langle 0,1,0\rangle = \hat j$

$\hat B(0) = \dfrac{\langle 0,0,2\rangle}{2} = \langle 0,0,1\rangle = \hat k$

$K(0) = \dfrac{2}{2^2} = \dfrac{1}{2} \to \rho(0) = 2$



osculating circle in xy plane
center (0,2,0)
radius 2

acceleration decomposition:

$$\vec a = \vec r\,'' = 2\langle 0,1,t\rangle$$

$$a_{\hat T} = \hat T\cdot\vec a = 2\frac{\langle 2,2t,t^2\rangle\cdot\langle 0,1,t\rangle}{t^2+2}$$

$$= \frac{2(2t+t^3)}{(t^2+2)} = \frac{2t(t^2+2)}{(t^2+2)} = 2t$$

$$a_{\hat N} = \hat N\cdot\vec a = 2\frac{\langle -2t, 2-t^2, 2t\rangle\cdot\langle 0,1,t\rangle}{(t^2+2)}$$

$$= \frac{2(2-t^2+2t^2)}{(t^2+2)} = \frac{2(t^2+2)}{(t^2+2)} = 2$$

$$\vec a = a_{\hat T}\hat T + a_{\hat N}\hat N = 2t\,\hat T + 2\,\hat N.$$