Find and classify critical points of $f(x,y) = \sin x + \sin y + \sin(x+y)$ on $0 \le x \le 2\pi$, $0 \le y \le 2\pi$

$f_x = \cos x + \cos(x+y) = 0 \xrightarrow{\text{expand}} \cos x + \cos x \cos y - \sin x \sin y = 0$

$f_y = \cos y + \cos(x+y) = 0$

subtract: $\cos x - \cos y = 0$

$\cos x = \cos y$

therefore $\sin y = \pm \sin x$

eliminate y in one equation

$\cos x + \cos x \cos x - \sin x(\pm \sin x) = 0$

$\cos x + \cos^2 x \mp \sin^2 x = 0$

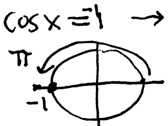
$\cos x + \cos^2 x \underbrace{\mp (1-\cos^2 x)}_{\mp 1 \pm \cos^2 x} = 0$

$(1 \pm 1)\cos^2 x + \cos x \mp 1 = 0$

lower sign: ($\sin y = -\sin x$) ($\cos y = \cos x$) reflected across horizontal axis on unit circle

$0 \cos^2 x + \cos x - 1 = 0$

$\cos x = -1 \rightarrow x = \pi$



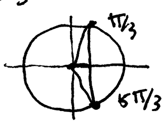
$\sin x = 0$

$\cos y = \cos x = -1 \rightarrow y = \pi$

(π, π)

upper sign: $\sin y = \sin x$, $\cos y = \cos x$

i.e. same point on unit circle: $(\cos x, \sin x) = (\cos y, \sin y)$



$x = \frac{\pi}{3}, \frac{5\pi}{3}$

$y = \frac{\pi}{3}, \frac{5\pi}{3}$

$2\cos^2 x + \cos x - 1 = 0$

$\cos x = \frac{-1 \pm \sqrt{1-4(2)(-1)}}{2(2)} = \frac{-1 \pm 3}{4}$

$= \frac{1}{2}, -1$

$\frac{1}{2}$: 60° reference angle. -1: $x = \pi = y$ as before

$(\frac{\pi}{3}, \frac{\pi}{3})$, $(\frac{5\pi}{3}, \frac{5\pi}{3})$

classify 3 critical pts

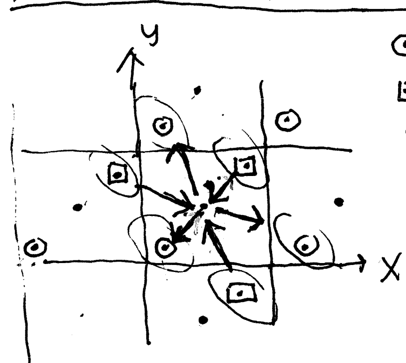
$f_{xx} = -\sin x - \sin(x+y)$

$f_{yy} = -\sin y - \sin(x+y)$

$f_{xy} = -\sin(x+y)$

	(π,π)	$(\frac{\pi}{3},\frac{\pi}{3})$	$(\frac{5\pi}{3},\frac{5\pi}{3})$
f_{xx}	$-\sin\pi - \sin 2\pi = 0$	$-\sin\frac{\pi}{3} - \sin\frac{2\pi}{3} = -2\left(\frac{\sqrt{3}}{2}\right) = -\sqrt{3} < 0$	$-\sin\frac{5\pi}{3} - \sin\frac{10\pi}{3} = \sqrt{3} > 0$
f_{yy}	$-\sin\pi - \sin 2\pi = 0$	$-\sin\frac{\pi}{3} - \sin\frac{2\pi}{3} = -2\left(\frac{\sqrt{3}}{2}\right) = -\sqrt{3} < 0$	$= \sqrt{3} > 0$
f_{xy}	$-\sin 2\pi = 0$	$-\sin\frac{2\pi}{3} = -\frac{\sqrt{3}}{2}$	$-\sin\frac{10\pi}{3} = \frac{\sqrt{3}}{2}$
$f_{xx}f_{yy} - f_{xy}^2$	0	$\sqrt{3}\sqrt{3} - \left(\frac{\sqrt{3}}{2}\right)^2 = 3 - \frac{3}{4} > 0$	$\sqrt{3}\sqrt{3} - \left(\frac{\sqrt{3}}{2}\right)^2 > 0$
	inconclusive, see Maple plots or:	local max	local min

periodic lattice of local max/mins, saddle pts:



⊙ local max, ⊡ local min } surrounded by nearby closed contours

• saddle pt → { 3 directions rise into saddle pt from the nearby 3 local mins (concave down); 3 directions rise from saddle pt towards the nearby 3 local maxs (concave up) }

This is therefore a saddle pt. *

* [at a differentiable critical point (hor. tan. plane) if values of function increase/decrease approaching the pt, curve must be concave down/up.]