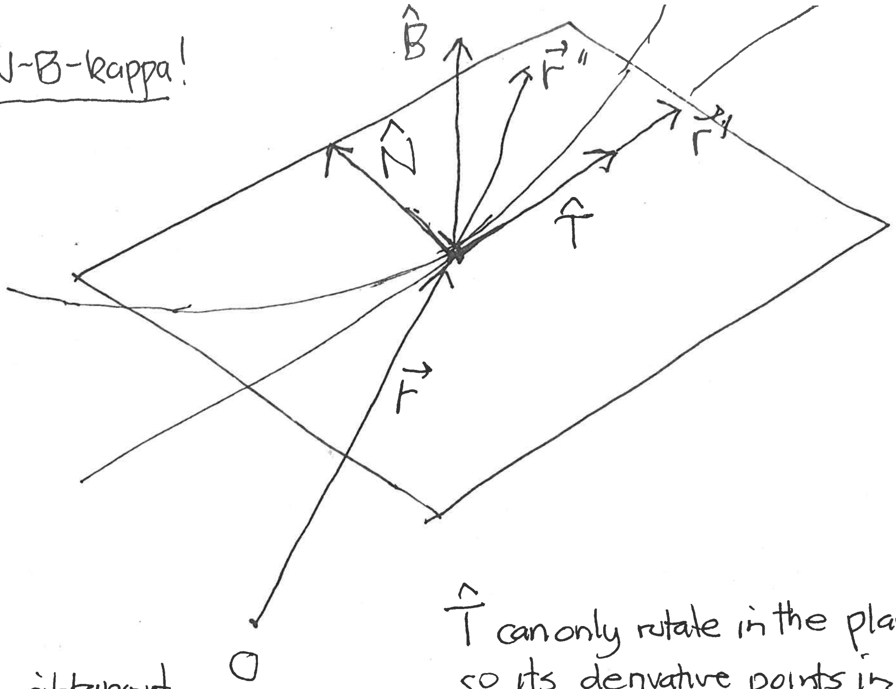
T-N-B-kappa!



$\hat{T}$ unittangent

$\hat{N}$ "normal"

$\hat{B}$ "binormal"

$\hat{T}-\hat{N}-\hat{B}$ triad

same cross-product relations as $\hat{\imath}, \hat{\jmath}, \hat{k} = \hat{\imath}\times\hat{\jmath}$.

$\hat{T}$ can only rotate in the plane of $\vec{r}\,', \vec{r}\,''$ so its derivative points in the orthogonal direction $\hat{N}$ on the "concave" side of the tangent line.

Define $\hat{B} = \hat{T}\times\hat{N}$, unit normal to the $\vec{r}\,', \vec{r}\,''$ plane

But $\vec{b} = \vec{r}\,'\times\vec{r}\,''$ is a normal to plane on same side so

$$\vec{B} = \hat{b} = \frac{\vec{r}\,'\times\vec{r}\,''}{|\vec{r}\,'\times\vec{r}\,''|}$$ easy!

$\hat{N} = \hat{B}\times\hat{T}$ ← easy!

$\hat{T}\,' \propto \underset{\text{direction of } \hat{T}'}{\hat{N}}$ so $\hat{N} = \dfrac{\hat{T}\,'}{|\hat{T}\,'|}$ ← hard, quotient rule.

$$\hat{T} = \frac{d\vec{r}}{ds} = \frac{dr/dt}{ds/dt} = \frac{\vec{r}\,'}{|\vec{r}\,'|}$$

only calculation needed to finish story

Finally curvature!

$$\frac{d\hat{T}}{ds} = \underbrace{\kappa}_{\text{curvature} \geq 0}\hat{N} = \frac{d\hat{T}/dt}{ds/dt} = \frac{\hat{T}\,'}{|\vec{r}\,'|} \to \kappa = \frac{|\hat{T}\,'|}{|\vec{r}\,'|} = \ldots = \frac{|\vec{r}\,'\times\vec{r}\,''|}{|\vec{r}\,'|^3}$$

$\frac{ds}{dt} = s' = |\vec{r}\,'| \to$

$$\vec{r}\,' = s'\hat{T}$$
$$\vec{r}\,'' = s''\hat{T} + s'\hat{T}\,'$$
$$\vec{r}\,'\times\vec{r}\,'' = (s'\hat{T})\times(\underbrace{s''\hat{T}}_{0} + s'\hat{T}\,') = (s')^2\,\hat{T}\times\underbrace{\hat{T}\,'}_{s'\kappa\hat{N}} = (s')^3\kappa\hat{N}$$

$$\left.\begin{aligned}\frac{d\hat{T}}{ds} &= \frac{d\hat{T}/dt}{ds/dt} = \frac{\hat{T}\,'}{s'}\\ &\equiv \kappa\hat{N}\end{aligned}\right\}\ \hat{T}\,' = \kappa s'\hat{N}$$

$$|\vec{r}\,'\times\vec{r}\,''| = (s')^3\kappa \to \kappa = \frac{|\vec{r}\,'\times\vec{r}\,''|}{(s')^3} = \frac{|\vec{r}\,'\times\vec{r}\,''|}{|\vec{r}\,'|^3}$$ ✓