

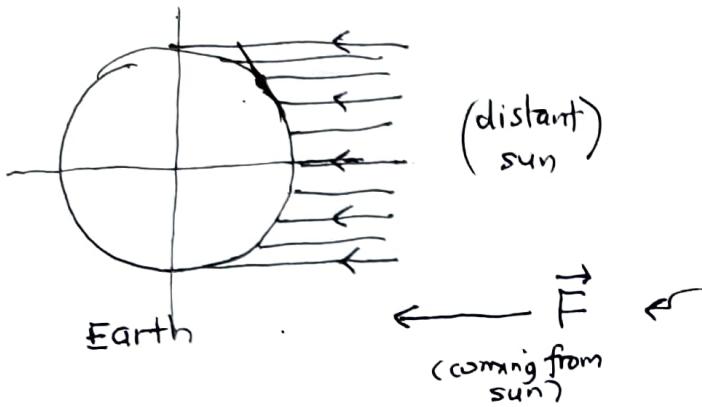
Everyday flux (surface) integral example

Energy from sun is delivered by parallel rays.

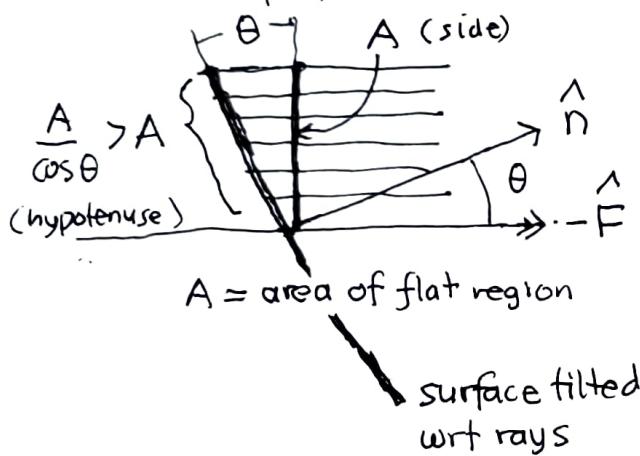
$$\text{power} = \text{energy} / \text{unit time}$$

Let P = power delivered per unit area perpendicular to the sun's rays (constant at Earth)

$$\text{Let } \vec{F} = P \hat{F} \quad (\text{energy flux})$$



cross-section perpendicular to surface



$PA = \text{power delivered to area } A \text{ per unit time}$

$$= P_0 \left(\frac{A}{\cos \theta} \right)$$

power per unit tilted area

same amount of power delivered to larger area

$$\text{so } P_0 = P \cos \theta$$

The amount of power delivered to tilted area is reduced by the cosine of the angle between $-\hat{F}$ and \hat{n} :

$$\cos \theta = (-\hat{F}) \cdot \hat{n}$$

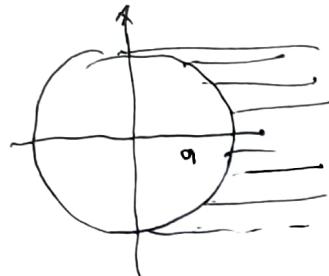
$$P_0 = P \cos \theta = (-P \hat{F}) \cdot \hat{n} = (-\vec{F}) \cdot \hat{n}$$

(projected along normal)

(16.5-9)

Surface & line integrals and Gauss & Stokes

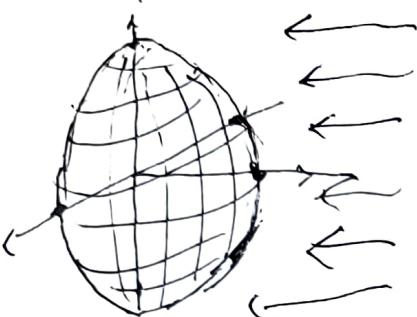
(2)



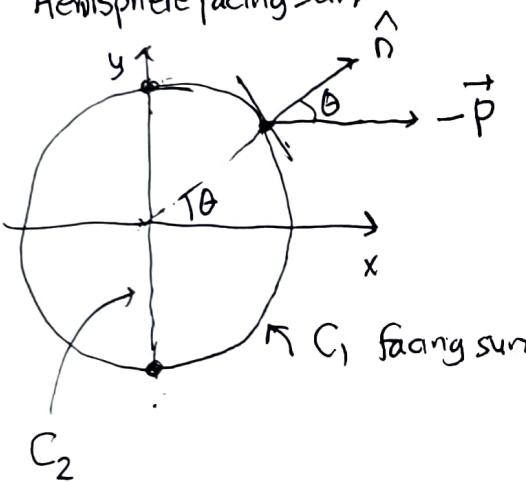
But this is only the plane cross-section thru the center of the spherical Earth in 3D space.

We must integrate up the normal component of the incoming power vector over the hemisphere facing the sun

If we lived in a 2D world we would integrate over the semicircular Earth boundary with respect to the arclength



Hemisphere facing Sun



$$G_1: \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{P \cos \theta}_{P_1} \underbrace{(a d\theta)}_{ds} = P a \sin \theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = P(2a)$$

$$G_2: \int_{-a}^a P dy = P y \Big|_{-a}^a = P(2a)$$

Same!

don't need calculus for this,
by definition just multiply
P by length perpendicular
to rays

i.e. the total amount of power delivered to the curved Earth surface is the same as delivered to a flat screen with the same profile.

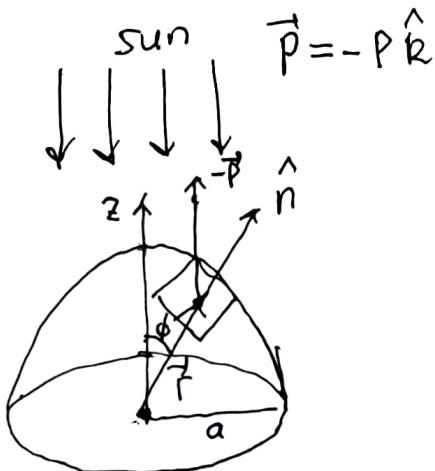
This is an example of integrating the normal component of a vector field (constant in this example) along a curve with respect to the arclength.

BUT the Earth is spherical!

16.6-9

surface & line integrals and Gauss & Stokes

③



let's let the rays come into the upper hemisphere along the z-axis and use spherical coords.

$$\vec{r} = \langle a \sin\phi \cos\theta, a \sin\phi \sin\theta, a \cos\phi \rangle$$

$$= a \underbrace{\langle \sin\phi \cos\theta, \sin\phi \sin\theta, \cos\phi \rangle}_{\vec{r} = \hat{n} \text{ outward normal}}$$

$\hat{n} \cdot (-\vec{p}) = p \cos\phi$ normal component

$$dV = dp \underbrace{(p^2 \sin\phi d\phi d\theta)}_{dS}$$

orthogonal coord system so ϕ, θ are orthogonal coordinates on the spherical surface $p=a$!

$$dS = p d\phi (r d\theta) = p^2 \sin\phi d\phi d\theta$$

$$= a^2 \sin\phi d\phi d\theta$$

differential of surface area easy!

$$S_{\text{sphere}} = \int_0^{2\pi} \int_0^{\pi} a^2 \sin\phi d\phi d\theta \quad (= \underset{\text{sphere}}{\iint} dS)$$

$$= a^2 \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi d\phi$$

$$= a^2 (2\pi) \underbrace{(-\cos\phi \Big|_0^{\pi})}_2 = 4\pi a^2$$

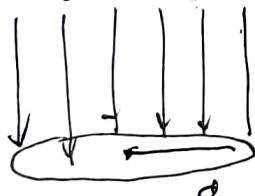
it works!

Integrate normal component over hemisphere $\phi=0..,\pi/2$:

$$\int_0^{2\pi} \int_0^{\frac{\pi}{2}} p \cos\phi a^2 \sin\phi d\phi d\theta = p a^2 \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \underbrace{\sin\phi}_{u} \underbrace{\frac{\cos\phi d\phi}{du}}_{\frac{\sin^2\phi}{2}} \rightarrow \frac{p a^2}{2}$$

$$= p (\pi a^2)$$

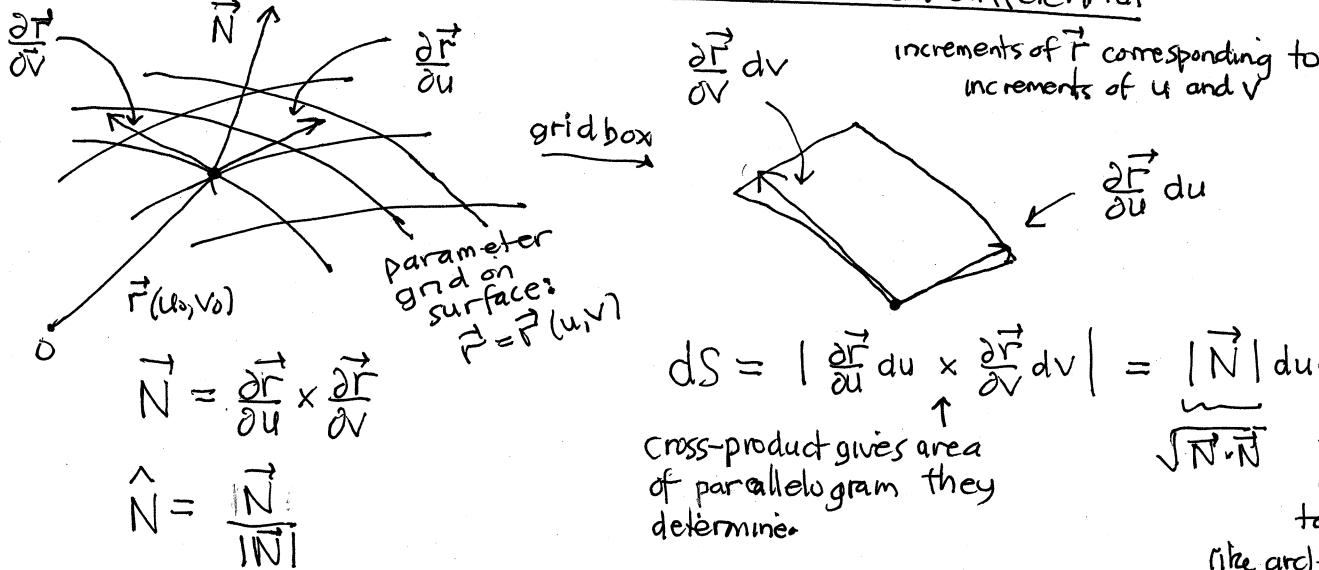
same as integral over plane cross-section
of rays hitting the Earth



$$\rightarrow A = \pi a^2$$

} just multiply P
by circular profile area!

surface integral: normal vector and surface area differential



surface integrals integrate the normal component of a vector field against the differential of surface area.

$$\iint_{\text{surface}} \vec{F}(\vec{r}(u, v)) \cdot \hat{N}(\vec{r}(u, v)) \, ds$$

$\underbrace{|\vec{N}(\vec{r}(u, v))|}_{\vec{N}(\vec{r}(u, v))} \, dudv$

$$= \iint_{u_1}^{u_2} \iint_{v_1}^{v_2} \vec{F}(\vec{r}(u, v)) \cdot \vec{N}(\vec{r}(u, v)) \, dv \, du$$

square root goes away
for vector integrals

easier than integrating scalars

$$\left[\int_C \vec{F}(\vec{r}(t)) \cdot \hat{T}(\vec{r}(t)) \, ds \right] \frac{|\vec{r}'(t)|}{|\vec{r}'(t)|} \, dt$$

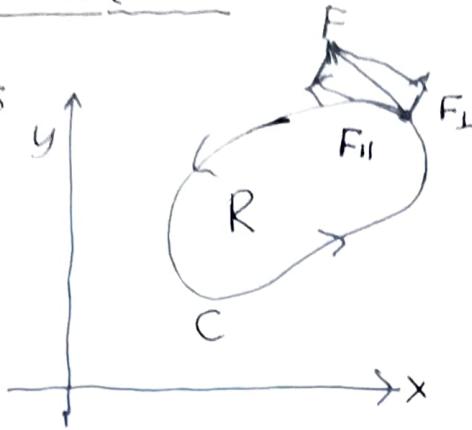
$$= \int_{t_1}^{t_2} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$$

just like for the line integral case

Green-Stokes-Gauss (8 flux)

16.6-9: 5

2-d Greens



$$F_{\perp} = \hat{n} \cdot \vec{F}$$

$$F_{\parallel} = \hat{T} \cdot \vec{F}$$

\hat{n} = outward normal

\hat{T} = counterclockwise unit normal

$$\hat{n} \times \hat{T} = \hat{k}$$

righthand rule!

Stokes version:

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C F_{\parallel} ds$$

circulation of \vec{F}
around C

$$= \iint_R (\operatorname{curl} \vec{F}) \cdot \hat{k} dA$$

$\underbrace{\iint_R}_{\text{"circulation density"}}$

Gauss version:

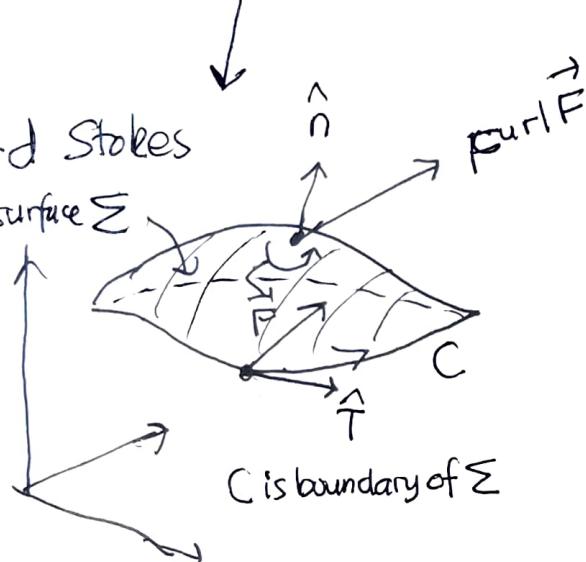
$$\oint_C F_{\perp} ds$$

"flux" of \vec{F}
out of R
thru C

$$= \iint_R (\operatorname{div} \vec{F}) dA$$

$\underbrace{\iint_R}_{\text{"flux density"}}$

3-d Stokes
surface Σ



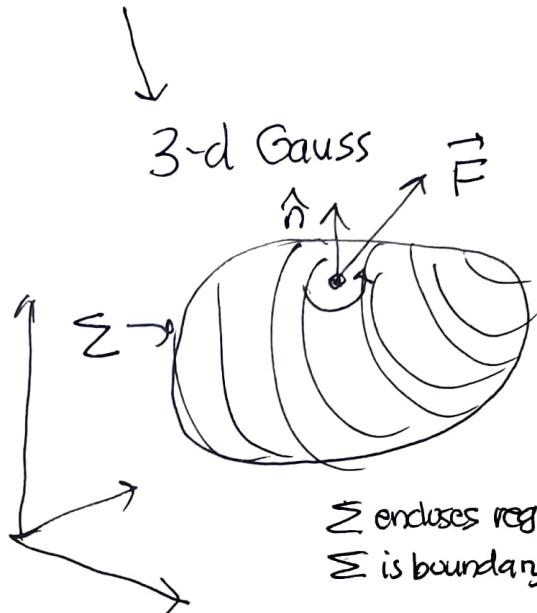
C is boundary of Σ

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C F_{\parallel} ds$$

$$= \iint_{\Sigma} (\operatorname{curl} \vec{F}) \cdot \hat{n} dS$$

$\underbrace{dS}_{\text{vector differential of surface area}}$

(flux of $\operatorname{curl} \vec{F}$
thru Σ in
direction \hat{n})



Σ encloses region R
 Σ is boundary of R

$$\iint_{\Sigma} \vec{F} \cdot d\vec{S} = \iint_{\Sigma} F_{\perp} dS$$

(flux of F thru
 Σ in
direction \hat{n})

$$= \iiint_R (\operatorname{div} \vec{F}) dV$$

in both cases:

integral of vector field on boundary = integral of its appropriate derivative on the interior region inside the boundary

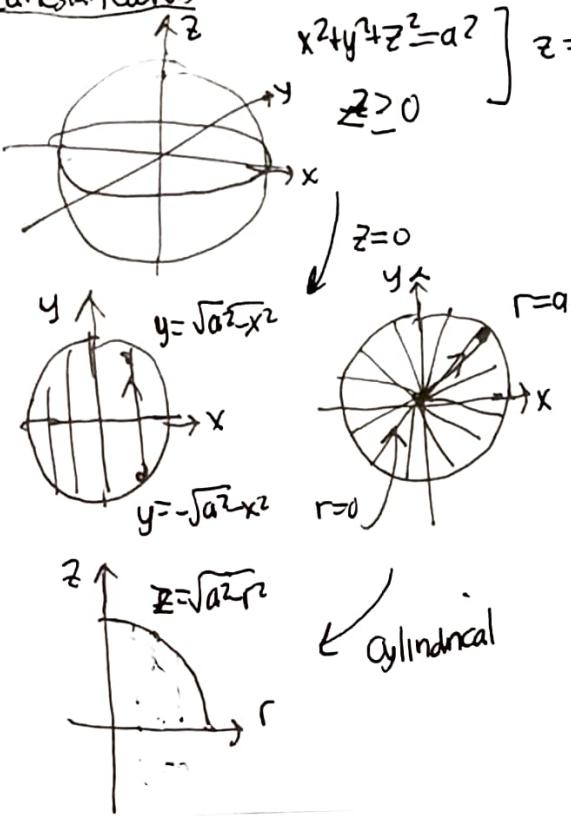
scalar surface integrals

Like scalar line integrals, sart factor makes them more difficult to evaluate

$$\iint_S f dS = \int_{U_1}^{U_2} \int_{V_1}^{V_2} f(\vec{r}(u, v)) \underbrace{\|\vec{N}(\vec{r}(u, v))\| du dv}_{\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right|} = \sqrt{\dots} \text{ sart factor}$$

$$\text{surface area of hemisphere} = 2\pi a^2 = \frac{1}{2}(4\pi a^2) = S$$

cartesian coords



cylindrical

spherical coords

$$\vec{r} = a \langle \sin\phi \cos\theta, \sin\phi \sin\theta, \cos\phi \rangle = a \hat{r}$$

$$\frac{\partial \vec{r}}{\partial \phi} = a \langle \cos\phi \cos\theta, \cos\phi \sin\theta, -\sin\phi \rangle$$

$$\frac{\partial \vec{r}}{\partial \theta} = a \langle -\sin\phi \sin\theta, \sin\phi \cos\theta, 0 \rangle$$

$$\frac{\partial \vec{r}}{\partial \phi} \times \frac{\partial \vec{r}}{\partial \theta} = \dots = a^2 \sin\phi \langle \sin\phi \cos\theta, \sin\phi \sin\theta, \cos\phi \rangle = a^2 \sin\phi \hat{r} \quad (\text{radial outward})$$

$$\left| \frac{\partial \vec{r}}{\partial \phi} \times \frac{\partial \vec{r}}{\partial \theta} \right| = a^2 \sin\phi$$

$$S = \int_0^{2\pi} \int_0^{\pi/2} a^2 \sin\phi \, d\phi \, d\theta = 2\pi a^2 \int_0^{\pi/2} \sin\phi \, d\phi = 2\pi a^2 \left[-\cos\phi \right]_0^{\pi/2} = 2\pi a^2$$

$$\frac{\partial \vec{r}}{\partial x} = \langle 1, 0, \frac{1}{z} \sqrt{a^2 - x^2 - y^2} \rangle = \langle 1, 0, -\frac{x}{z} \rangle$$

$$\frac{\partial \vec{r}}{\partial y} = \langle 0, 1, -\frac{y}{z} \rangle$$

$$\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = \dots = \langle \frac{x}{z}, \frac{y}{z}, 1 \rangle = \frac{\langle x, y, z \rangle}{z} = \frac{\vec{r}}{z}$$

radially outward!

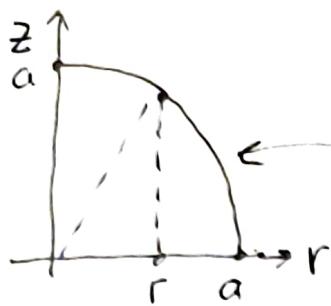
$$\left| \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right| = \frac{\sqrt{x^2 + y^2 + z^2}}{z} = \frac{a}{z} = \frac{a}{\sqrt{a^2 - x^2 - y^2}}$$

$$S = \underbrace{\int_{-a}^a \int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}}}_{\int_0^{2\pi} \int_0^a} dy \, dx \quad \begin{matrix} \text{hard} \\ \text{u-sub!} \end{matrix}$$

$$\underbrace{\int_0^a}_{\sqrt{a^2 - x^2 - y^2}} \underbrace{\int_0^{2\pi} r dr d\theta}_{\text{easier}} = 2\pi a^2 \checkmark$$

scalar surface integrals (2)

upper hemisphere (surface) in cylindrical coords.



$$x^2 + y^2 + z^2 = r^2 + z^2 = a^2 \Rightarrow z = \sqrt{a^2 - r^2}$$

$$\vec{r} = \langle r\cos\theta, r\sin\theta, \sqrt{a^2 - r^2} \rangle$$

$$r = 0..a, \theta = 0..2\pi$$

$$\frac{\partial \vec{r}}{\partial r} = \langle \cos\theta, \sin\theta, \frac{-r}{\sqrt{a^2 - r^2}} \rangle$$

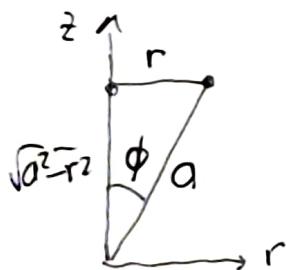
$$\frac{\partial \vec{r}}{\partial \theta} = \langle -r\sin\theta, r\cos\theta, 0 \rangle$$

$$\frac{\partial \vec{r}}{\partial r} \times \frac{\partial \vec{r}}{\partial \theta} = \dots = r \left\langle \frac{r\cos\theta}{\sqrt{a^2 - r^2}}, \frac{r\sin\theta}{\sqrt{a^2 - r^2}}, 1 \right\rangle = \frac{r}{\sqrt{a^2 - r^2}} \underbrace{\langle r\cos\theta, r\sin\theta, \sqrt{a^2 - r^2} \rangle}_{\vec{r} \text{ radially outward}}$$

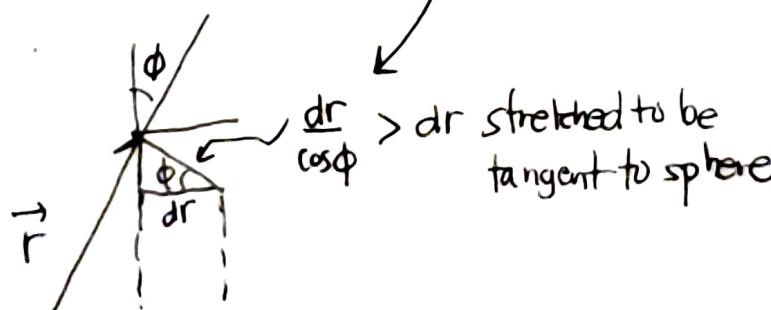
$$|\vec{r}| = a$$

$$\left| \frac{\partial \vec{r}}{\partial r} \times \frac{\partial \vec{r}}{\partial \theta} \right| = \frac{ar}{\sqrt{a^2 - r^2}} \rightarrow dS = \frac{ar}{\sqrt{a^2 - r^2}} dr d\theta$$

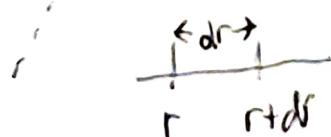
same as re-expressing
Cartesian integral in
polar coords!



$$\frac{a}{\sqrt{a^2 - r^2}} = \cos\phi \rightarrow dS = \left(\frac{dr}{\cos\phi} \right) (rd\theta)$$



$\frac{dr}{\cos\phi} > dr$ stretched to be tangent to sphere



16.6-9

surface & line integrals and Gauss & Stokes

16.6-9: 8

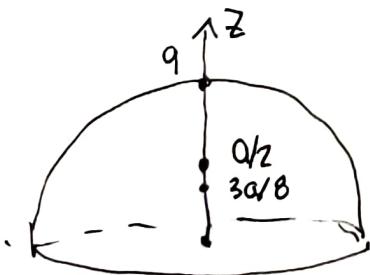
scalar surface integrals (3)

Compare centroid of hemispherical surface to hemispherical solid
Use spherical coords. Must be on z-axis by symmetry.

$$\bar{z}_{\text{Hsurface}} = \frac{\iint_S z \, dS}{\iint_S 1 \, dS} = \frac{\int_0^{2\pi} \int_0^{\pi/2} (a \cos\phi)(a^2 \sin\phi) \, d\phi \, d\theta}{\int_0^{2\pi} \int_0^{\pi/2} \sin\phi \cos\phi \, d\phi \, d\theta} \cdot \frac{2\pi a^2}{2\pi} = \frac{a}{2} \text{ halfway point!}$$

$$\bar{z}_{\text{Hsolid}} = \frac{\iiint_V z \, dV}{\iiint_V 1 \, dV} = \frac{\int_0^{2\pi} \int_0^{\pi/2} \int_0^a (\rho \cos\phi)(\rho^2 \sin\phi) \, d\rho \, d\phi \, d\theta}{\int_0^{2\pi} \int_0^{\pi/2} \int_0^a \sin\phi \cos\phi \, d\phi \, d\theta \cdot \frac{4\pi a^3/3}{\rho^4/4}} = \frac{\pi a^4/4}{2\pi a^3/3} = \frac{3}{8}a < \frac{1}{2}a \quad \text{lower}$$

(interior volume favors lower part of hemisphere, pulls down centroid)



$$\rho = a, \theta = 0..2\pi, \phi = 0.. \pi/2 \quad (\text{face})$$

$$\rho = a, \theta = 0..2\pi, \phi = 0.. \pi/2 \quad (\text{solid})$$