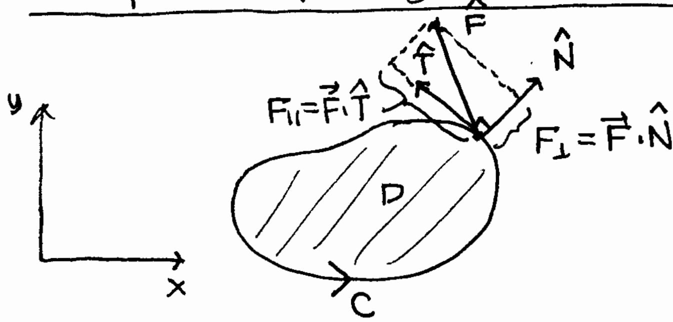
interpretation of divergence and curl (2d)



C: counterclockwise loop with interior D

$\hat{T}$: unit counterclockwise tangent

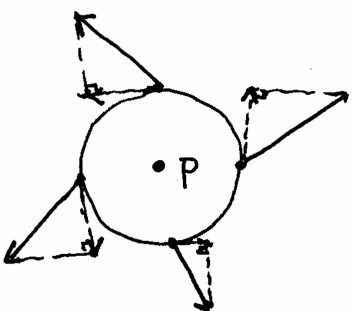
$\hat{N}$: unit outer normal

$F_{\parallel} = \vec{F} \cdot \hat{T}$ tangential component along curve

$F_{\perp} = \vec{F} \cdot \hat{N}$ normal component along curve

Green-Stokes

$$\oint_C \vec{F} \cdot \hat{T}\, ds = \text{circulation of } \vec{F} \text{ (counterclockwise) around } C = \iint_D \operatorname{curl}(\vec{F})_z\, dA$$



To interpret the value of $\operatorname{curl}(F)_z$ at a point P, shrink a small loop (circle or rectangle) down around P until the value of $\operatorname{curl}(\vec{F})_z$ across the loop is almost constant (within some tolerance).

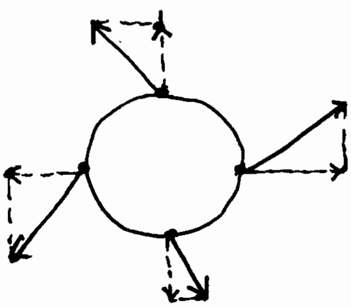
$$\approx \underbrace{\operatorname{curl}(\vec{F})_z \Big|_P}_{} \cdot \operatorname{area}(D)$$

only $F_{\parallel}$ contributes to the circulation around a loop

if nonzero there must always be a net circulation around the loop proportional to the area of the loop, in the counterclockwise sense if positive, clockwise if negative.

Green-Gauss

$$\oint_C \vec{F} \cdot \hat{N}\, ds = \text{net flux of } \vec{F} \text{ out of loop } C = \iint_D \operatorname{div}(\vec{F})\, dA$$



Again shrink a loop down around P until the value of $\operatorname{div}(\vec{F})$ is almost constant across the loop

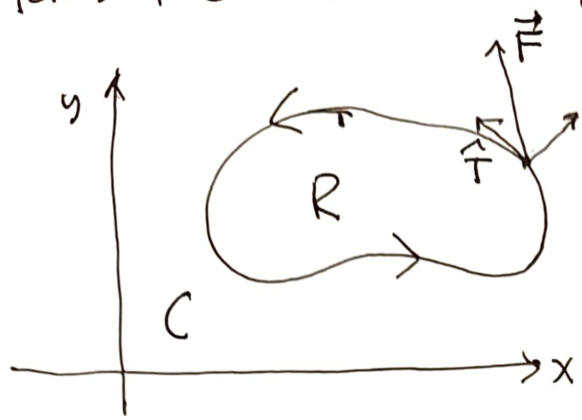
$$\approx \underbrace{\operatorname{div}(F)\Big|_P}_{} \cdot \operatorname{area}(D)$$

only $F_{\perp}$ contributes to the flux in or out of the loop

if nonzero there must always be a net flux in (if negative) or out (if positive) of the loop

grad, div, curl stuff: what does it mean?

2 forms of Green's theorem in plane:



Green Stokes

$$\underbrace{\oint_C \vec{F}\cdot\hat{T}\,ds}_{\text{total "circulation" of } \vec{F} \text{ around } C} = \iint_R \underbrace{(\text{curl}\,\vec{F})_3}_{\text{"circulation" density}}\,dA$$

Green-Gauss

$$\underbrace{\oint_C \vec{F}\cdot\hat{N}\,ds}_{\text{net "flux" out of } R \text{ of } \vec{F}} = \iint_R \underbrace{\text{div}\,\vec{F}}_{\text{"flux density"}}\,dA$$

examples to visualize

$$\vec{F} = \langle -y^2, x\rangle \quad (\text{curl}\,\vec{F})_3 = \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y^2) = 1 + 2y = \begin{cases} 1 & y=0 \\ 0 & y=-\frac{1}{2} \\ -1 & y=-1 \end{cases}$$

should show positive, zero, negative (clockwise) circulation around loops

$$\vec{F} = \langle x, y^2\rangle \quad \text{div}\,\vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y^2) = 1 + 2y = \Big\{ \text{ ditto}$$

should show positive, zero, negative divergence in pattern inside loops, all 3 signs for flux outward

But $\quad$ derivative $\Big(\underbrace{\vec{F}(\vec{r}) - \vec{F}(\vec{r}_0)}_{=0 \text{ at } \vec{r}=\vec{r}_0}\Big) = \text{derivative}\big(\vec{F}(\vec{r})\big)$ ← pattern easier to see

since curl/div only measure variation in pattern as x, y change

(see Maple worksheet)