

Grad, Curl, Div stuff (1) (vector language rules!)

notation: gradient $f \rightarrow \vec{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \equiv \underbrace{\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle}_{} f$

"del" is a vector derivative operator on scalars

$$\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \quad \text{applied to a function } f(x, y, z)$$

on its right like in 1-d:

$\frac{d}{dx} (\dots)$ acts on $f(x)$ to its right.

combining $\vec{\nabla}$ with the two vector operations "dot" and "cross" yields two vector operators on vector fields.

$$\text{grad } f = \vec{\nabla} f$$

$$\text{div } \vec{F} = \vec{\nabla} \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle F_1, F_2, F_3 \rangle = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\text{curl } \vec{F} = \vec{\nabla} \times \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle F_1, F_2, F_3 \rangle \quad (\text{clock permutations})$$

$$= \left\langle \frac{\partial F_3 - \partial F_2}{\partial y}, \frac{\partial F_1 - \partial F_3}{\partial z}, \frac{\partial F_2 - \partial F_1}{\partial x} \right\rangle$$

all act from left to right on expressions (scalar/vector)

We can only combine these in succession in the combinations

$$\text{div grad } f = \vec{\nabla} \cdot \vec{\nabla} f = \vec{\nabla}^2 f = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial z^2} \right\rangle = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \quad \text{"Laplacian"}$$

$$\text{curl grad } \vec{F} = \vec{\nabla} \times \vec{\nabla} f = 0 \quad \left. \begin{array}{l} \text{identically zero for differentiable} \\ \text{scalar, vector fields} \end{array} \right\}$$

$$\text{div curl } \vec{F} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0$$

nextpage!

→ If $\vec{F} = \vec{\nabla} f$ is a conservative vector field then

$$\vec{\nabla} \times \vec{F} = \left\langle \underbrace{\frac{\partial(\partial f)}{\partial y} - \frac{\partial(\partial f)}{\partial z}}_{=0}, \underbrace{\frac{\partial(\partial f)}{\partial z} - \frac{\partial(\partial f)}{\partial x}}_{=0}, \underbrace{\frac{\partial(\partial f)}{\partial x} - \frac{\partial(\partial f)}{\partial y}}_{=0} \right\rangle = 0$$

conversely if this condition is satisfied in a simply connected open set of space, a scalar potential exists.

scalar potentials easier to deal with than vector fields! energy versus force — physically important

ELECTRIC
GRAVITY
FIELDS
etc

Schroedinger's equation underlying all of CHEMISTRY!

16.5 grad, curl, div stuff

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example: inverse square force field

$$f = \frac{k}{|\vec{r}|}$$

scalar potential
function

$$\vec{F} = \nabla f = -\frac{k}{|\vec{r}|^2} \hat{r} = -\frac{k \vec{r}}{|\vec{r}|^3} = -\frac{k \langle x_1, y_1, z_1 \rangle}{(x^2 + y^2 + z^2)^{3/2}}$$

force
vector field

divergence:

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\frac{\partial F_1}{\partial x} = \frac{\partial}{\partial x} \left(\frac{-kx}{(x^2 + y^2 + z^2)^{3/2}} \right) = -k \left(\frac{(x^2 + y^2 + z^2)^{3/2} (1) - x \left(\frac{3}{2}\right)(...)^{1/2}(2x)}{(...)^3} \right)$$

$$= -k(x^2 + y^2 + z^2)^{1/2} \left(\frac{x^2 + y^2 + z^2 - 3x^2}{(x^2 + y^2 + z^2)^3} \right) = -k \frac{(-2x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{5/2}}$$

$$= -k \frac{x^2 - 2y^2 + z^2}{(x^2 + y^2 + z^2)^{5/2}}$$

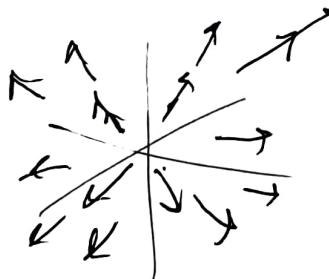
$$\frac{\partial F_2}{\partial y} = \dots$$

$$\frac{\partial F_3}{\partial z} = \dots$$

$$= -\frac{k(x^2 + y^2 - 2z^2)}{(x^2 + y^2 + z^2)^{5/2}}$$

$$\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = -k \frac{(-2x^2 + y^2 + z^2) + x^2 - 2y^2 + z^2 + x^2 + y^2 - 2z^2}{(x^2 + y^2 + z^2)^{5/2}} = 0 \quad \vec{F} \neq \langle 0, 0, 0 \rangle \text{ undefined !!}$$

$k > 0$: \vec{F} radial outward from origin, grows in length towards origin



the pattern of arrows "diverges" from origin
= location of point charge if electric field.

"infinite" divergence at origin but zero
everywhere else.

If $k > 0$: \vec{F} radial towards origin "converges" towards origin
(negative divergence)

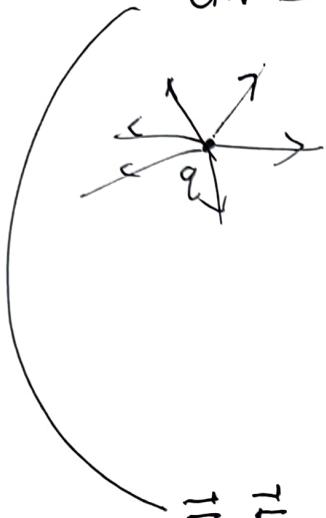
If electric field \vec{E} , then $\operatorname{div} \vec{E} = 0$ (Maxwell eqn)!
(wait for explanation)

16.5 grad, div, curl stuff

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Maxwell eqn for static electric field:

$$\operatorname{div} \vec{E} = 4\pi\rho \quad (\text{charge density})$$



point charge has total charge q
concentrated at a point
so $\rho \rightarrow \infty$ at that point

$\operatorname{div} \vec{E} = 0$ everywhere except at pt charge
but remnant of ∞ density at pt is
diverging electric field vectors and
flow lines

$$\vec{\nabla} \cdot \vec{E} = \nabla \cdot (\vec{\nabla} f) = \vec{\nabla}^2 f = 4\pi\rho$$

determines electric potential
in general — Laplaces eqn.

existence of potential due to

$$\nabla \times \vec{E} = \vec{\nabla} \times \vec{\nabla} f \equiv 0 \quad \text{allows } \vec{E} \text{ to be "scalar" represented by its potential}$$

static magnetic field: $\operatorname{div} \vec{B} = 0 \leftarrow (4\pi \underbrace{\rho_{\text{mag}}}_{=0})$

$$\text{Let } \vec{B} = \operatorname{curl} \vec{A} = \vec{\nabla} \times \vec{A}$$

magnetic charges
don't exist.

$$\text{Then } \operatorname{div} \vec{B} = \operatorname{div} \operatorname{curl} \vec{A} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) \equiv 0$$

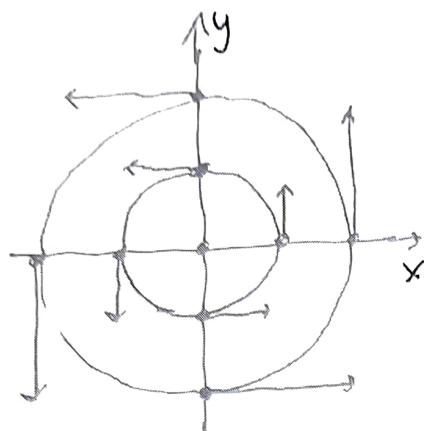
solves Maxwell eqn for static magnetic field.

16.5 Grad, Curl, Div stuff

④

examples of purely curling / purely diverging vector fields

pure
curl
field
(zero
(divergence))



$$\vec{F} = \langle -y, x \rangle \quad \text{tangential to circles:}$$

$$= \hat{r} \times \vec{r} \quad \vec{F} \cdot \langle x, y \rangle = 0$$

$$\hat{r} \cdot \text{curl } \vec{F} = \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y) = 2 \text{ constant}$$

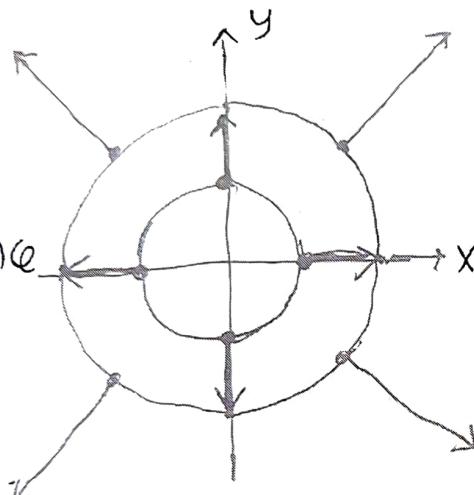
$$\text{div } \vec{F} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0$$

"circulating pattern"

(flip sign,
clockwise)

moving around origin
counterclockwise

pure
divergence
field
(zero
(curl))



$$\vec{F} = \langle x, y \rangle = \vec{r}$$

$$\hat{r} \cdot \text{curl } \vec{F} = \frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x) = 0$$

$$\text{div } \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 2 \text{ constant}$$

"divergent" spray of vectors

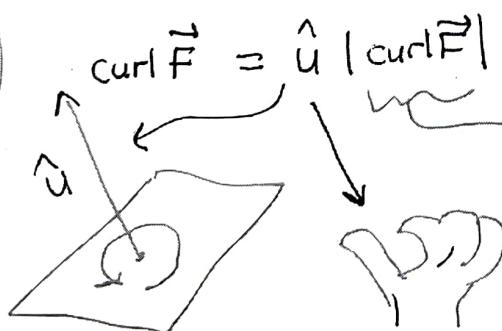
coming out of origin

"source"

(flip sign, goes into origin)

"sink"

3d
curl



$\text{curl } \vec{F} = \hat{u} |\text{curl } \vec{F}|$
magnitude measures circulation in
plane orthogonal to \hat{u}

right hand rule
gives direction
of circulation about direction \hat{u}

16.5 grad, curl, div stuff

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Rewrite Green's Theorem in plane

We can use curl and divergence by defining:

$$\vec{F}(x, y, z) = \underbrace{\langle F_1(x, y), F_2(x, y), 0 \rangle}_{\text{vector field in plane thought of as a horizontal vector field independent of } z}$$

vector field in plane thought of as a horizontal vector field independent of z

$$\operatorname{curl} \vec{F} = \vec{\nabla} \times \vec{F} = \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle$$

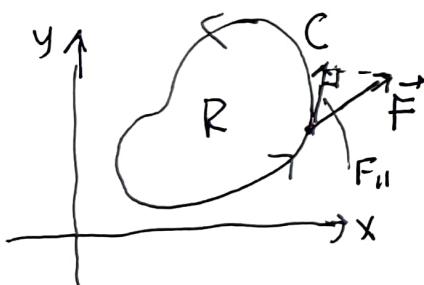
(1) (2) (3)

$$= \langle 0, 0, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \rangle = \hat{k} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$$\hat{k} \cdot (\vec{\nabla} \times \vec{F}) = \hat{k} \cdot \operatorname{curl} \vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \quad \text{integrand of Green's Thm}$$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} dA = \iint_R \hat{k} \cdot \operatorname{curl} \vec{F} dA$$

↑ simple closed curve.



interpretation
integrate F_1 along C
"total circulation"
of \vec{F} around C

"density"
for circulation
at each point
integrate up
to get
total circulation.

$$\boxed{\int_C (\vec{F} \cdot \hat{T}) ds = \iint_R \hat{k} \cdot \operatorname{curl} \vec{F} dA}$$

"Stokes'"
version of
Greens Thm

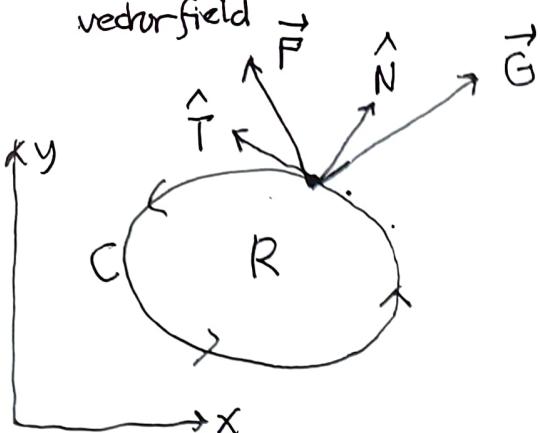
↑ single integral ↑ double integral

inner integration undoes
vector derivative
to result in outer integral
of \vec{F} itself.

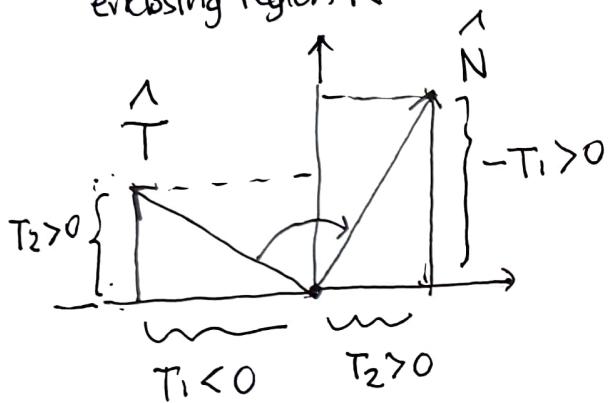
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We can re-express Green's Thm in terms of the "normal" component of a vector field



Simple closed loop curve
counterclockwise orientation
enclosing region R



Green's Thm integrates
 $F_{\parallel} = \vec{F} \cdot \hat{T}$ along C wrt arclength

The outward normal \hat{N} pointing away from R is rotated 90° clockwise from \hat{T}

Rotate \vec{F} to \vec{G} by same rotation.

Re-express $\vec{F} \cdot \hat{T}$ in terms of $\vec{G} \cdot \hat{N}$
as follows:

$$\text{so } \hat{N} = \langle T_2, -T_1 \rangle \rightarrow \\ \text{repeat with } \vec{F} \text{ to get } \vec{G} \\ \vec{G} = \langle F_2, -F_1 \rangle = \langle G_1, G_2 \rangle$$

$$\text{so } \vec{G} \cdot \hat{N} = \langle F_2, -F_1 \rangle \cdot \langle T_2, -T_1 \rangle \\ = F_2(T_2) + F_1(-T_1) = \vec{F} \cdot \hat{T}$$

dot product does not change under rotation

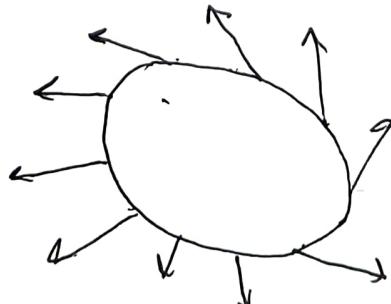
Note: $F_1 = -G_2$, $F_2 = G_1$

$$\frac{\partial F_2 - \partial F_1}{\partial x - \partial y} = \frac{\partial (G_1)}{\partial x} - \frac{\partial (-G_2)}{\partial y} = \frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} = \operatorname{div} \vec{G}$$

$$\text{so } \oint_C \vec{F} \cdot \hat{T} ds = \iint_R \underbrace{\frac{\partial F_2 - \partial F_1}{\partial x - \partial y}}_{\operatorname{div} \vec{G}} dA$$

$$\boxed{\oint_C \vec{G} \cdot \hat{N} ds = \iint_R \operatorname{div} \vec{G} dA} \quad \begin{array}{l} (\text{true for any } \vec{G}) \\ (\text{can restate with } \vec{F}) \end{array}$$

outward
normal component



measures "flux" of vectors outward thru bounding curve