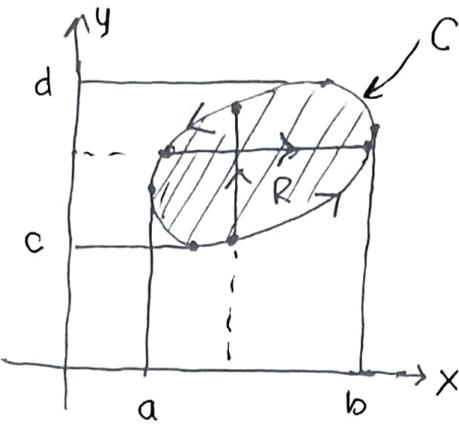


Green's Theorem (1)

16.4: 1

Consider a region R of the plane that can be integrated either as y -first then x or as x -first then y as a single integral. Its boundary is a "simple closed curve" C .

orient C by the counterclockwise direction.



$$y = g_1(x) \dots g_2(x) \quad \text{while } x = a \dots b$$

[OR]

$$x = h_1(y) \dots h_2(y) \quad \text{while } y = c \dots d$$

Green's Thm:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} dA = \iint_R \underbrace{\frac{\partial F_2}{\partial x}}_{\text{"under each other"!}} dA - \iint_R \underbrace{\frac{\partial F_1}{\partial y}}_{\text{"under each other"!}} dA$$

Note: if $\vec{F} = \vec{\nabla}f$, then the integrand is identically zero so $\oint_C \vec{F} \cdot d\vec{r} = 0$, which must be true for any conservative vector field.

Green's Thm is useful for general vector fields (nonconservative).

B
I
G

P
I
C

T
U
R
E

"S undoes d" (single integrals)

$$\int_a^b \frac{df}{dt} dt = \int_a^b df = f \Big|_a^b = \Delta f$$

$$\oint_C \vec{\nabla}f \cdot d\vec{r} = \int_a^b \frac{df(\vec{r}(t))}{dt} dt = \int_a^b df(\vec{r}(t)) \Big|_a^b = f(\vec{r}(b)) - f(\vec{r}(a))$$

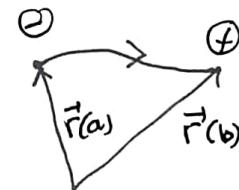
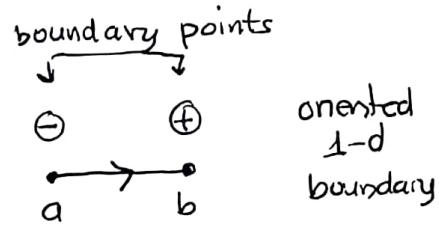
integrates integrand to boundary

(double integrals)

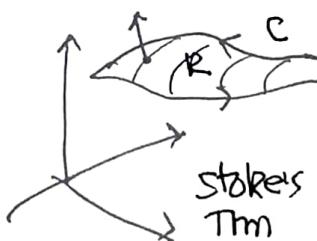
$$\iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} dA = \oint_C \vec{F} \cdot d\vec{r}$$

(triple integrals... etc)

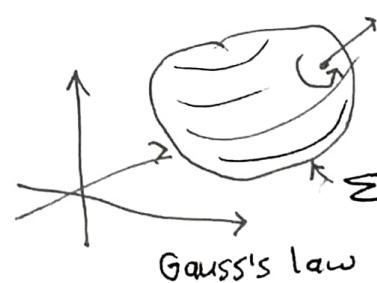
innermost integration undoes derivative, remaining integrals take place on boundary



oriented
2-d
boundary



Stokes' Thm



Gauss's law

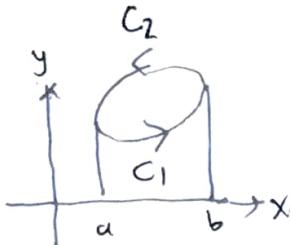
Green's Theorem (2)

"proof"

16.4: 2

$$\oint_C \vec{F} \cdot d\vec{r} = \int_C \langle F_1, F_2 \rangle \cdot \langle dx, dy \rangle = \int_C F_1 dx + F_2 dy$$

$$= \int_C F_1 dx + \int_C F_2 dy$$



$$C_1: x=t, y=g_1(t), t=a..b, dx=dt$$

$$C_2: x=t, y=g_2(t), t=b..a, dx=dt$$

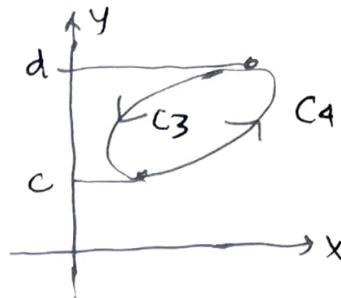
opp. directions

$$\int_C F_1 dx = \int_a^b F_1(t, g_1(t)) dt \quad ①$$

$$- \int_a^b F_1(t, g_2(t)) dt \quad ②$$

$$+ \int_C F_2 dy = \int_c^d F_2(h_2(t), t) dt \quad ③$$

$$- \int_c^d F_2(h_1(t), t) dt \quad ④$$

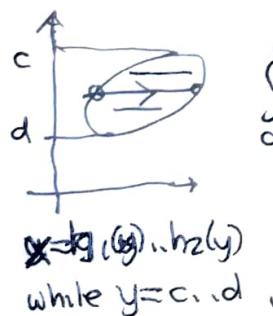


$$C_3: y=t, x=h_1(y), t=d..c, dy=dt$$

$$C_4: y=t, x=h_2(y), t=c..d, dy=dt$$

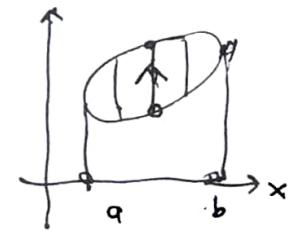
opp. directions

$$\iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} dA = \iint_R \frac{\partial F_2}{\partial x} dA - \iint_R \frac{\partial F_1}{\partial y} dA$$



$$\iint_R \frac{\partial F_2}{\partial x} dA = \iint_R \frac{\partial F_2}{\partial x} dx dy - \iint_R \frac{\partial F_2}{\partial x} dx dy$$

$\underbrace{\int_c^d \int_{h_1(y)}^{h_2(y)} \frac{\partial F_2}{\partial x}(x, y) dx dy}_{F_2(x, y) \Big|_{x=h_1(y)}^{x=h_2(y)}} - \iint_R \frac{\partial F_2}{\partial x} dx dy$



$$\iint_R \frac{\partial F_2}{\partial x} dx dy = \int_c^d \int_a^b F_2(h_2(y), y) dy dx - \left(\int_a^b F_1(x, g_2(x)) dx - \int_a^b F_1(x, g_1(x)) dx \right)$$

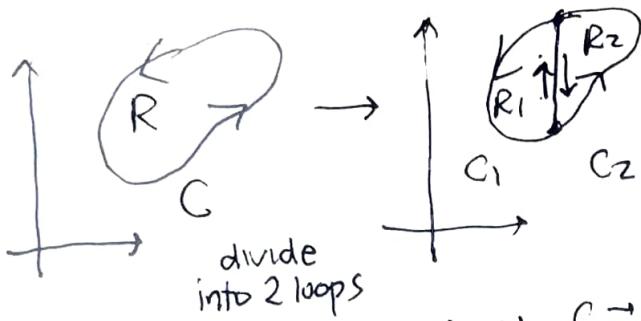
signs all match!

\therefore (the integral = double integral) Q.E.D.

Integration undoes differentiation! (when cleverly arranged)

Greens' theorem (3)

generalize integration region



counter-clockwise orientation always

$$\oint_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

since contributions from overlapping internal boundary curves cancel (opposite orientations)

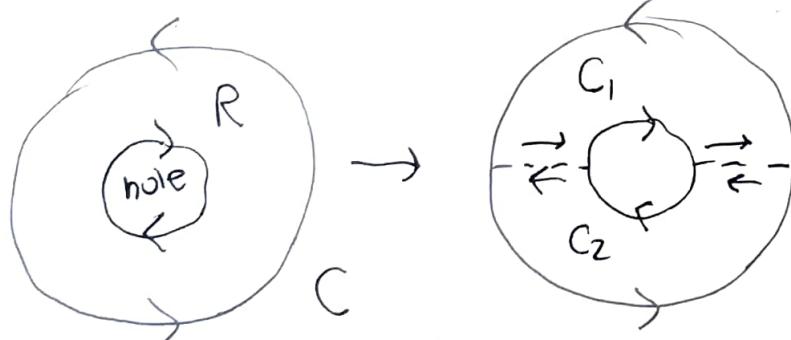
$$= \iint_{R_1} \frac{\partial F_2}{\partial x} \frac{\partial F_1}{\partial y} dA + \iint_{R_2} \frac{\partial F_2}{\partial x} \frac{\partial F_1}{\partial y} dA$$

Green's thm applies to each separately

$$= \iint_R \frac{\partial F_2 - \partial F_1}{\partial x - \partial y} dA$$

combine integrals

so any interior of a simple closed curve can be broken up into subregions which allow integration in either order, so above proof extends to the more general region

Add holes to the mix

sum 2 contributions from $C_1 \cup C_2$ where Green's Thm (simple) applies
but requires clockwise orientation on internal boundaries

$$= C_{out} + C_{in}$$

\curvearrowleft orientations opposite

This generates the difference of the two counter-clockwise line integrals

OR!

$$\begin{aligned} \iint_R \dots dA &= \iint_{R+Rinner} \dots dA - \iint_{Rinner} \dots dA \\ &= \oint_{C_{outer}} \vec{F} \cdot d\vec{r} - \oint_{C_{inner}} \vec{F} \cdot d\vec{r} \end{aligned}$$

Green's Theorem (4)

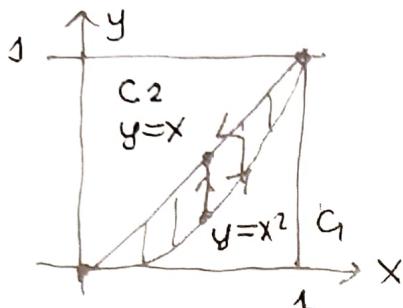
enough talk - worked example:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} dA$$

16.4: 4

$$\vec{F}(x,y) = \langle y^2, x^2 \rangle$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(y^2) = 2x - 2y = 2(x-y)$$



$$\begin{aligned} \iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} dA &= \int_0^1 \int_{x^2}^x 2(x-y) dy dx \\ &= \dots = \boxed{\frac{1}{30}} \quad \text{easy by hand, easier with technology} \end{aligned}$$

$$C_1: \vec{r}(t) = \langle t, t^2 \rangle, t = 0..1 \quad (+), \quad \vec{r}'(t) = \langle 1, 2t \rangle, \quad \vec{F}(\vec{r}(t)) = \langle (t^2)^2, t^2 \rangle$$

$$C_2: \vec{r}(t) = \langle t, t \rangle, t = 1..0 \quad (-), \quad \vec{r}'(t) = \langle 1, 1 \rangle, \quad \vec{F}(\vec{r}(t)) = \langle t^2, t^2 \rangle$$

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^1 \underbrace{\langle t^4, t^2 \rangle}_{t^4 + 2t^3} \cdot \langle 1, 2t \rangle dt = \left. \frac{t^5}{5} + \frac{2t^4}{4} \right|_0^1 = \frac{1}{5} + \frac{1}{2}$$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_1^0 \underbrace{\langle t^2, t^2 \rangle}_{2t^2} \cdot \langle 1, 1 \rangle dt = - \left. \frac{2t^3}{3} \right|_1^0 = -\frac{2}{3}$$

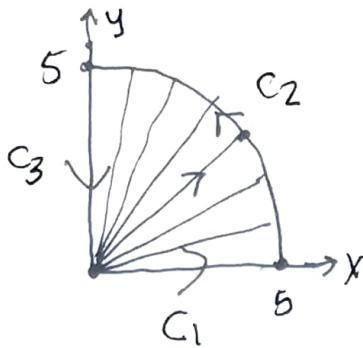
$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} = \left(\frac{1}{5} + \frac{1}{2} \right) - \frac{2}{3} = \frac{7}{10} - \frac{2}{3} = \frac{21-20}{30} = \boxed{\frac{1}{30}}$$

Green's Theorem (5)

16.4: 5

example:

$$x^2 + y^2 = 25 \quad \iint_R f dA = \int_0^{\frac{\pi}{2}} \int_0^5 f(r, \theta) r dr d\theta$$



$$\vec{F} = \langle \sin x, \sin y + xy^2 + \frac{1}{3}x^3 \rangle$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial x} (\sin y + xy^2 + \frac{1}{3}x^3) - \frac{\partial}{\partial y} (\sin x) \\ = y^2 + x^2 = r^2 !$$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{\frac{\pi}{2}} \int_0^5 (r^2) r dr d\theta$$

$$\downarrow \quad \frac{r^4}{4} \Big|_0^5 \cdot \int_0^{\frac{\pi}{2}} d\theta = \left(\frac{\pi}{2}\right)\left(\frac{5^4}{4}\right)$$

$$= \boxed{\frac{59\pi}{8}} \text{ easy!}$$

$$C_1: \vec{r}(t) = \langle t, 0 \rangle, t = 0..5$$

$$C_3: \vec{r}(t) = \langle 0, t \rangle, t = 5..0$$

$$C_2: \vec{r}(t) = \langle 5 \cos t, 5 \sin t \rangle, t = 0.. \frac{\pi}{2}$$

$$C_1: \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = \langle \sin t, 0 \rangle \cdot \langle 1, 0 \rangle = \sin t \quad \int_0^5 \sin t dt = -\cos t \Big|_0^5 = 1 - \cos 5$$

$$C_3: \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = \langle 0, \sin t \rangle \cdot \langle 0, 1 \rangle = \sin t \quad \int_5^0 \sin t dt = \dots = \frac{-(1 - \cos 5)}{\cancel{cancel}}$$

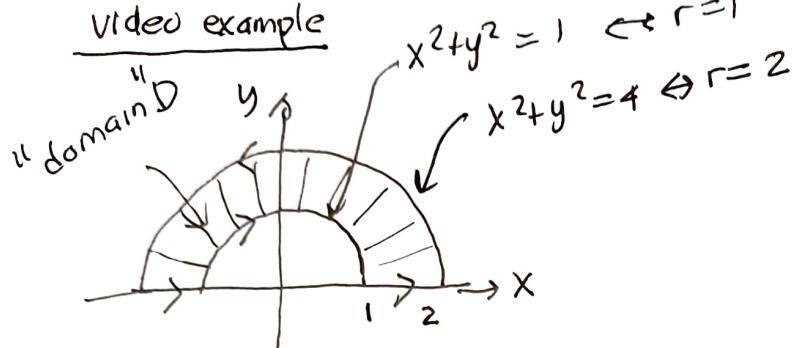
$$C_2: \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = \underbrace{\langle \sin(5 \cos t), \sin(5 \sin t) + (5 \cos t)(5 \sin t)^2 + \frac{1}{3}(5 \cos t)^3 \rangle}_{\text{u-sub}} \cdot \langle -5 \sin t, 5 \cos t \rangle \\ = \sin(5 \cos t) (-5 \sin t) + \sin(5 \sin t) (5 \cos t) \\ + (5 \cos t)^2 (5 \sin t)^2 + \frac{1}{3} (5 \cos t)^4$$

... u-sub + powers of sines, cosines so doable

result is $\boxed{\frac{59\pi}{8}}$ ✓

16.4

(6)

Video example

$r=1..2$ while $\theta = 0..\pi$
 (dearly calls for polar coord integration)

 C : counterclockwise loop

4 separate segments -

line integral

$$\int_C \vec{F} \cdot d\vec{r}$$

ugly!

lot of hard work.

$$\vec{F}(x,y) = \langle x e^{-2x}, x^4 + 2x^2 y^2 \rangle$$

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x} (x^4 + 2x^2 y^2) = 4x^3 + 4x y^2 = 4x(x^2 + y^2)$$

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y} (x e^{-2x}) = 0$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 4x(x^2 + y^2) = 4(r \cos \theta) r^2 = 4r^3 \cos \theta$$

$$\int \vec{F} \cdot d\vec{r} = \int_0^\pi \int_1^2 (4r^3 \cos \theta) r dr d\theta$$

$$= \int_0^\pi \int_1^2 4r^4 \cos \theta dr d\theta \xrightarrow{\text{factors}} \underbrace{4 \int_0^\pi \cos \theta d\theta}_{\sin \theta \Big|_0^\pi} \underbrace{\int_1^2 r^4 dr}_{\frac{r^5}{5} \Big|_1^2} = 0$$

done!

= 0!