16.2b

0a

Vector line integral notation

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \langle F_1, F_2, F_3\rangle \cdot \langle dx, dy, dz\rangle$$
$$= \underbrace{\int_C F_1\,dx + F_2\,dy + F_3\,dz}$$

when a line integral is represented symbolically in this "multiplied out" form of the dot product, simply identify the coefficients of the differentials as the vector field components

If given a parametrized curve segment
$\vec{r} = \vec{r}(t)$, $t = a..b$
then simply use the formula

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}(\vec{r}(t))\cdot d\vec{r}(t) = \int_a^b F(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$$

If given a curve, come up with a parametrization and use this formula.

From the additivity of integrals:

$$\int_C \vec{F}\cdot d\vec{r} = \int_C F_1\,dx + F_2\,dy + F_3\,dz = \underbrace{\int_C F_1\,dx} + \underbrace{\int_C F_2\,dy} + \underbrace{\int_C F_3\,dz}$$

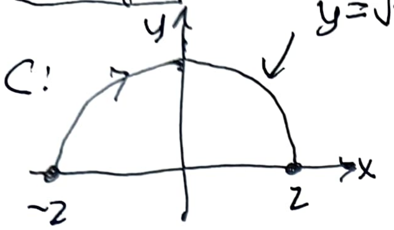
each is a vector line integral of a vector field with only one nonzero component, so are handled exact like the general case.

16.2b Vector line integral notation (Ob)

When parametrizing a curve it is not necessary to rename any variable to be t.

Example

$y=\sqrt{4-x^2}$ for $-2\le x\le 2$.



$F(x,y) = \langle -y, 2x\rangle$

let $t=x=-2..2$

then $y=\sqrt{4-t^2}$

$\vec{r}(t) = \langle t, \sqrt{4-t^2}\rangle$

$$\int_C \vec{F}\cdot d\vec{r} = \int \underbrace{\vec{F}(\vec{r}(t))}_{\langle -\sqrt{4-t^2},\, 2t\rangle}\cdot \underbrace{\vec{r}\,'(t)}_{\langle 1, -\frac{t}{\sqrt{4-t^2}}\rangle}\,dt$$

$$=\int_{-2}^{2}\left(-\sqrt{4-t^2}-\frac{2t^2}{\sqrt{4-t^2}}\right)dt$$

OR

keep x as parameter.

then $dy = \frac{dy}{dx}dx = \frac{1}{2}\frac{-2x\,dx}{\sqrt{4-x^2}} = \frac{-x\,dx}{\sqrt{4-x^2}}$

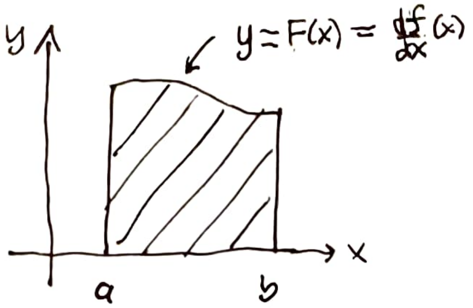
$$\int_C \vec{F}\cdot d\vec{r} = \int_C -y\,dx + 2x\,dy$$

$$=\int_{-2}^{2} -(\sqrt{4-x^2})\,dx + 2x\left(\frac{-x\,dx}{\sqrt{4-x^2}}\right)$$

$$=\int_{-2}^{2}\left(-\sqrt{4-x^2}-\frac{x(2x)}{\sqrt{4-x^2}}\right)dx$$

same result

Higher Dimensional Generalization of Fundamental Theorem of Calculus



Fundamental Theorem of Calculus:

$$\int_a^b F(x)\,dx = \int_a^b \frac{df(x)}{dx}\,dx = f(x)\Big|_a^b$$
$$= f(b) - f(a)$$

definite integrals are evaluated using antidifferentiation

f = antiderivative of F

HIGHER DIMENSION

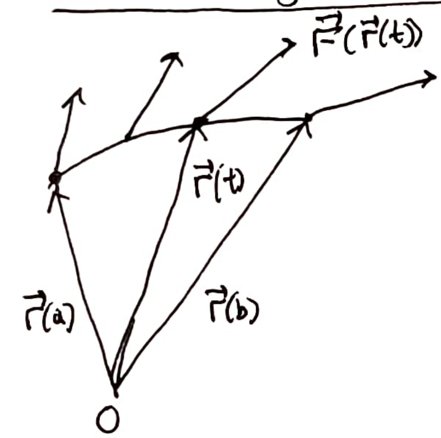
scalar field $f(\vec{r}) \xrightarrow{\text{differentiate}} \vec{\nabla} f(\vec{r}) \equiv \vec{F}(\vec{r})$ gradient vector field

antidifferentiation

If we start with $\vec{F}(\vec{r})$, how do we know if it is the gradient of a function? LATER...

called "conservative vector field", $f(\vec{r})$ is called its "potential function"

vector line integral of conservative vector field



$$\vec{F}(\vec{r}) = \left\langle \frac{\partial f}{\partial x}(\vec{r}), \frac{\partial f}{\partial y}(\vec{r}), \frac{\partial f}{\partial z}(\vec{r}) \right\rangle$$
$$\vec{F}(\vec{r}(t)) = \left\langle \frac{\partial f}{\partial x}(\vec{r}(t)), \frac{\partial f}{\partial y}(\vec{r}(t)), \frac{\partial f}{\partial z}(\vec{r}(t)) \right\rangle$$
$$\vec{r}\,'(t) = \left\langle \frac{dx}{dt}(t), \frac{dy}{dt}(t), \frac{dz}{dt}(t) \right\rangle$$
$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = \frac{\partial f}{\partial x}(\vec{r}(t))\frac{dx}{dt}(t) + \frac{\partial f}{\partial y}(\vec{r}(t))\frac{dy}{dt}(t) + \frac{\partial f}{\partial z}(\vec{r}(t))\frac{dz}{dt}(t)$$
$$= \frac{df(\vec{r}(t))}{dt} \qquad \text{(CHAIN RULE)}$$

C: $\vec{r} = \vec{r}(t)$, $t = a..b$

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt = \int_a^b \frac{d}{dt} f(\vec{r}(t))\,dt = f(\vec{r}(t))\Big|_a^b$$
$$= f(\vec{r}(b)) - f(\vec{r}(a))$$

final − initial

$$\vec{F}\cdot dr = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$
$= df$ is an "exact" differential

> The line integral of a conservative vector field is just the difference in its potential function between the initial and terminal point.

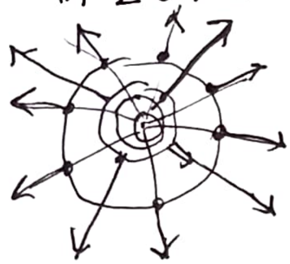
Higher Dim Fund Theorem & Conservative Vector Fields (2)

Example: inverse square force field

potential function $f = \dfrac{k}{|\vec{r}|} \longrightarrow \vec{F} = \vec{\nabla} f = \dfrac{-k\hat{r}}{|\vec{r}|^2}$ force field

level curves/surfaces are concentric circles / spheres about origin in 2-d / 3-d

points in radial direction towards ($k>0$) away from ($k<0$) the origin

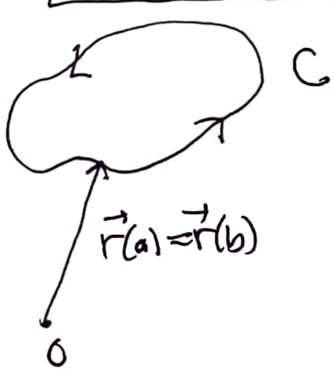


undefined at origin! domain omits origin! (location of point charge or mass)

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = f(\vec{r}(b)) - f(\vec{r}(a)) = \Delta f(\vec{r})\Big|_a^b$$

$$= \frac{k}{|\vec{r}(b)|} - \frac{k}{|\vec{r}(a)|}$$ independent of "path"

$\boxed{\text{conservative vector fields have path independent line integrals}}$



on a simple closed curve (starting pt = ending pt)

$$\int_C \vec{F} \cdot d\vec{r} = 0$$

"O" $\longrightarrow \oint_C \vec{F} \cdot d\vec{r} \leftarrow$ notation for line integral around closed loop

"simple" means no self-intersections

$\boxed{\text{conservative vector fields have vanishing line integral around any simple closed curve.}}$

is the converse statement true? It depends on the details

Higher Dim Fund Theorem & Conservative Vector Fields (3)

2-d case: $\vec{F} = \vec{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle F_1, F_2 \rangle$

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= F_1 \\ \frac{\partial f}{\partial y} &= F_2 \end{aligned}\right\}$$

If we start with $\vec{F}$ and want to "antidifferentiate" to go back to f which we dont know if it exists, we must solve this pair of partial differential equations for the unknown solution function $f(x,y)$

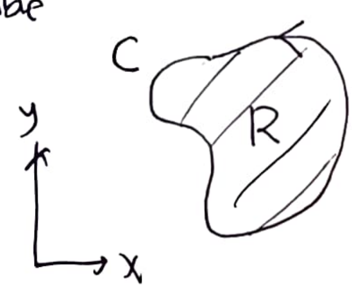
$\downarrow$ consequence (necessary)

$$\left.\begin{aligned} \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) &= \frac{\partial F_1}{\partial y} \\ \| \\ \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) &= \frac{\partial F_2}{\partial x} \end{aligned}\right\} \longrightarrow \boxed{\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}} \text{ or } \boxed{\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0}$$

order does not matter if f differentiable

Suppose $\vec{F}$ satisfies this condition on a region of ~~space~~ the plane (2-d case), does that guarantee a solution f exists?

FACT



If path C encloses a "simply connected" (no holes) region, where this condition is satisfied everywhere, then $\vec{F}$ does have a potential f.

But how to find?

2 methods exist to find f.

a) successive line integrals from reference point

b) solve PDEs

Higher Dim Fund Theorem & Conservative Vector Fields (4)

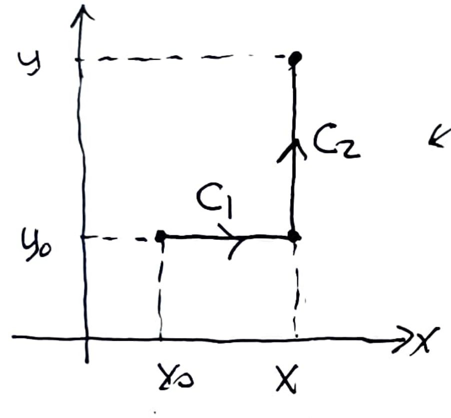
a) successive line integration

1-d case: $\int f(x)\,dx = F(x) + \underset{\text{"constant"}}{\mathscr{C}}$

Define $F(x) = \underbrace{\int_a^x f(u)\,du}$ $\rightarrow$ $F'(x) = f(x)$!

integral formula for an antiderivative
if can find way to evaluate this formula, it gives us a particular antiderivative

2-d case:



since line integral should not depend on the path, choose this path from (x_0, y_0) to (x, y):

$$\underbrace{f(\vec{r}) - f(\vec{r_0})}_{\mathscr{C}} = \int_C \vec{F}\cdot d\vec{r}$$

$$= \int_{C_1} F_1\,dx + \int_{C_2} F_2\,dy$$

$$= \underbrace{\int_{x_0}^{x} F_1(t, y_0)\,dt + \int_{y_0}^{y} F_2(x, t)\,dt}$$

yeah, sure, in principle but easier by method b)!

C_1: $y = y_0$ while $x = x_0 .. x$

$\vec{r} = \langle t, y_0 \rangle$, $t = x_0 .. x$ $(dy = 0)$

C_2: "$x = x$" while $y = y_0 .. y$

$\vec{r} = \langle x, t \rangle$, $t = y_0 .. y$ $(dx = 0)$

$\vec{r}\,' = \langle 1, 0 \rangle$ $\quad \vec{F}\cdot\vec{r}\,' = F_1$

$\vec{r}\,' = \langle 0, 1 \rangle$ $\quad \vec{F}\cdot\vec{r}\,' = F_2$

b) solve PDE's: solve one by partial integration, plug result into other, then solve second equation.

example. $\vec{F} = \langle 3 + 2xy,\ x^2 - 3y^2 \rangle \rightarrow \frac{\partial}{\partial x}(x^2 - 3y^2) - \frac{\partial}{\partial y}(3 + 2xy) = 2x - 2x = 0$ ✓

$\frac{\partial f}{\partial x} = 3 + 2xy \xrightarrow{\int dx} \int \underbrace{\frac{\partial f}{\partial x}\,dx}_{\text{"}f} = \int 3 + 2xy\,dx = 3x + x^2y + \underbrace{C(y)}$

"constant" of integration can depend on other indiv. var.

$\frac{\partial f}{\partial y} = x^2 - 3y^2 \longleftarrow$ so $f = 3x + x^2y + C(y)$

$\hookrightarrow \underbrace{\frac{\partial}{\partial y}(3x + x^2y + C(y))}_{= 0 + x^2 + C'(y)} = x^2 - 3y^2 \Big\}$ $C'(y) = -3y^2$ (no x!)

$C(y) = \int -3y^2\,dy = -y^3 + k$

so $f = 3x + x^2y - y^3 + k$ done.

Higher Dim Fund Theorem & Conservative Vector Fields (5)

$\boxed{\text{3-d case}}$ $\quad \vec{F} = \langle F_1, F_2, F_3 \rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$

$\frac{\partial f}{\partial x} = F_1$: $\quad \frac{\partial^2 f}{\partial y \partial x} = \partial F_1/\partial y$, $\quad \frac{\partial^2 f}{\partial z \partial x} = \partial F_1/\partial z$

$\frac{\partial f}{\partial y} = F_2$: $\quad \frac{\partial^2 f}{\partial x \partial y} = \partial F_2/\partial x$, $\quad \frac{\partial^2 f}{\partial z \partial y} = \partial F_2/\partial z$

$\frac{\partial f}{\partial z} = F_3$: $\quad \frac{\partial^2 f}{\partial x \partial z} = \partial F_3/\partial x$, $\quad \frac{\partial^2 f}{\partial y \partial z} = \partial F_3/\partial y$

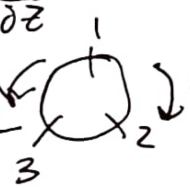
equal in pairs!

$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0 \quad \leftarrow$ 2-d condition if $F_3 \equiv 0$

$\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} = 0$

$\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} = 0$

3 conditions



clock arithmetic

$+ \quad \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} = 0$

permute $(1,2,3)$

where $(x_1, x_2, x_3) = (x, y, z)$

Example

$\vec{F} = \langle e^{xz} yz,\ e^{xz},\ e^{xz} xy \rangle = \langle F_1, F_2, F_3 \rangle$

$$\begin{cases} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial x}(e^{xz}) - \frac{\partial}{\partial y}(e^{xz} yz) = z e^{xz} - z e^{xz} = 0 \checkmark \\ \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} = \frac{\partial}{\partial y}(e^{xz} xy) - \frac{\partial}{\partial z}(e^{xz}) = x e^{xz} - x e^{xz} = 0 \checkmark \\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} = \frac{\partial}{\partial z}(e^{xz} yz) - \frac{\partial}{\partial x}(e^{xz} xy) = xyz e^{xz} + y e^{xz} - (xyz e^{xz} + y e^{xz}) \\ \qquad = 0 \checkmark \end{cases}$$

guarantees soln exists. solve 3 PDEs in any order.

$\int \left[\frac{\partial f}{\partial x} = yz e^{xz} \right] dx \to f = \int yz e^{xz} dx = y\cancel{z} \frac{e^{xz}}{\cancel{z}} + C(y,z) = y e^{xz} + C(y,z)$

$\frac{\partial f}{\partial y} = e^{xz} \quad = \quad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(y e^{xz} + C(y,z)\right) = e^{xz} + \frac{\partial C}{\partial y}(y,z)$

$0 = \frac{\partial C}{\partial y}(y,z) \xrightarrow{\int dy} C(y,z) = C(z)$

indof y!

so $f = y e^{xz} + C(z)$

$\frac{\partial f}{\partial z} = xy e^{xz} \quad \frac{\partial f}{\partial z} = xy e^{xz} + C'(z)$

$0 = C'(z) \xrightarrow{\int dz} C(z) = k$

if terms had not cancelled, successive eqns would have been inconsistent!

so $f = y e^{xz} + k \checkmark$

Higher Dim Fund Theorem & Conservative Vector Fields (6)

Counter-example

$\vec{F} = \langle F_1, F_2, F_3 \rangle = \langle y, z, x \rangle$

$\int \left[\frac{\partial f}{\partial x} = y \right] dx \longrightarrow f = \int y\,dx = xy + C(y,z)$

$\frac{\partial f}{\partial y} = z$ —— $\frac{\partial f}{\partial y} = x + C(y,z) \longrightarrow z = x + C(y,z)$

$C(y,z) = z - x$

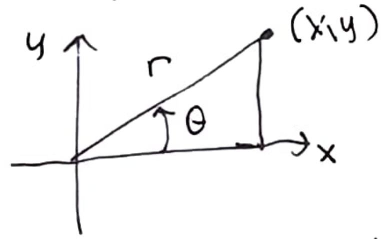
$\frac{\partial f}{\partial z} = x$

no x on LHS

makes no sense, inconsistent eqns

no soln.

example



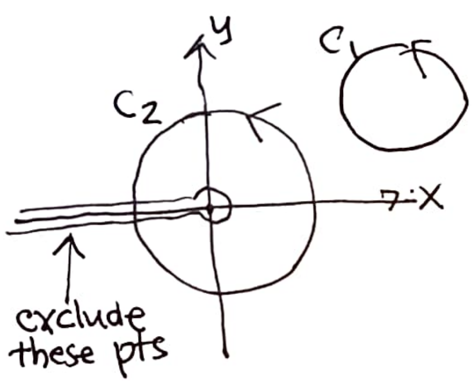
$$\tan\theta = \frac{y}{x} \longrightarrow \theta = \arctan(y,x) = \begin{cases} \arctan(y/x), & x > 0 \quad \text{(quads 1,4)} \\ \arctan(y/x) + \pi, & x<0, y>0 \quad \text{(quad2)} \\ \arctan(y/x) - \pi, & x<0, y<0 \quad \text{(quad3)} \\ \frac{\pi}{2}, & x=0, y>0 \\ -\frac{\pi}{2}, & x=0, y<0 \\ \text{undefined at } x=0=y \end{cases}$$

continuous and differentiable everywhere except on negative x-axis —
jump discontinuity: $\Delta\theta = 2\pi$!

$$d\theta = \frac{\partial \theta}{\partial x}dx + \frac{\partial \theta}{\partial y}dy = \frac{1}{1+(y/x)^2}\left(-\frac{y}{x^2}\right)dx + \frac{1}{1+(y/x)^2}\left(\frac{1}{x}\right)dy$$

$$= \frac{-y\,dx + x\,dy}{x^2+y^2} = \underbrace{\frac{\langle -y, x\rangle}{x^2+y^2}}\cdot\langle dx, dy\rangle$$

$\vec{F} \equiv \vec{\nabla}\theta$ so $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$ since $\vec{F}$ is a gradient



For any loop that does not cross "bad points" the line integral vanishes

But any loop passing thru "bad points" has nonzero value $2\pi = \Delta\theta$ if in counterclockwise direction

$\oint_{C_1} d\theta = 0, \quad \oint_{C_2} d\theta = \Delta\theta = 2\pi$